

# $L_\infty$ -algebra connections and applications to String- and Chern-Simons $n$ -transport

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February 15, 2008

## Abstract

We give a generalization of the notion of a Cartan-Ehresmann connection from Lie algebras to  $L_\infty$ -algebras and use it to study the obstruction theory of lifts through higher String-like extensions of Lie algebras. We find (generalized) Chern-Simons and BF-theory functionals this way and describe aspects of their parallel transport and quantization.

It is known that over a D-brane the Kalb-Ramond background field of the string restricts to a 2-bundle with connection (a gerbe) which can be seen as the obstruction to lifting the  $PU(H)$ -bundle on the D-brane to a  $U(H)$ -bundle. We discuss how this phenomenon generalizes from the ordinary central extension  $U(1) \rightarrow U(H) \rightarrow PU(H)$  to higher categorical central extensions, like the String-extension  $\mathbf{BU}(1) \rightarrow \text{String}(G) \rightarrow G$ . Here the obstruction to the lift is a 3-bundle with connection (a 2-gerbe): the Chern-Simons 3-bundle classified by the first Pontrjagin class. For  $G = \text{Spin}(n)$  this obstructs the existence of a String-structure. We discuss how to describe this obstruction problem in terms of Lie  $n$ -algebras and their corresponding categorified Cartan-Ehresmann connections. Generalizations even beyond String-extensions are then straightforward. For  $G = \text{Spin}(n)$  the next step is “Fivebrane structures” whose existence is obstructed by certain generalized Chern-Simons 7-bundles classified by the second Pontrjagin class.

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# 1 Introduction

The study of extended  $n$ -dimensional relativistic objects which arise in string theory has shown that these couple to background fields which can naturally be thought of as  $n$ -fold categorified generalizations of fiber bundles with connection. These structures, or various incarnations of certain special cases of them, are probably most commonly known as (bundle-) $(n-1)$ -gerbes with connection. These are known to be equivalently described by Deligne cohomology, by abelian gerbes with connection (“and curving”) and by Cheeger-Simons differential characters. Following [6, 9] we address them as  $n$ -bundles with connection.

fundamental object	background field
$n$ -particle	$n$ -bundle
$(n-1)$ -brane	$(n-1)$ -gerbe

Table 1: **The two schools of counting** higher dimensional structures. Here  $n$  is in  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

In string theory, the first departure from bundles with connections to higher bundles with connection occurred with the fundamental (super)string coupling to the Neveu-Schwarz (NS)  $B$ -field. Locally, the  $B$ -field is just an  $\mathbb{R}$ -valued two-form. However, the study of the path integral, which amounts to ‘exponentiation’, reveals that the  $B$ -field can be thought of as an abelian gerbe with connection whose curving corresponds to the  $H$ -field  $H_3$  or as a Cheeger-Simons differential character, whose holonomy [34] can be described [20] in the language of bundle gerbes [62].

The next step up occurs with the M-theory (super)membrane which couples to the  $C$ -field [11]. In supergravity, this is viewed locally as an  $\mathbb{R}$ -valued differential three-form. However, the study of the path integral has shown that this field is quantized in a rather nontrivial way [75]. This makes the  $C$ -field not precisely a 2-gerbe or degree 3 Cheeger-Simons differential character but rather a shifted version [28] that can also be modeled using the Hopkins-Singer description of differential characters [46]. Some aspects of the description in terms of Deligne cohomology is given in [26].

From a purely formal point of view, the need of higher connections for the description of higher dimensional branes is not a surprise:  $n$ -fold categorified bundles with connection should be precisely those objects that allow us to define a consistent assignment of “phases” to  $n$ -dimensional paths in their base space. We address such an assignment as **parallel  $n$ -transport**. This is in fact essentially the definition of Cheeger-Simons differential characters [25] as these are consistent assignments of phases to chains. However, abelian bundle gerbes, Deligne cohomology and Cheeger-Simons differential characters all have one major restriction: they only know about assignments of elements in  $U(1)$ .

While the group of phases that enter the path integral is usually abelian, more general  $n$ -transport is important nevertheless. For instance, the latter plays a role at intermediate stages. This is well understood for  $n = 2$ : over a  $D$ -brane the abelian bundle gerbe corresponding to the NS field has the special property that it measures the obstruction to lifting a  $PU(H)$ -bundle to a  $U(H)$ -bundle, i.e. lifting a bundle with structure group the infinite projective unitary group on a Hilbert space  $H$  to the corresponding unitary group [15] [16]. Hence, while itself an abelian 2-structure, it is crucially related to a nonabelian 1-structure.

That this phenomenon deserves special attention becomes clear when we move up the dimensional ladder: The Green-Schwarz anomaly cancelation [40] in the heterotic string leads to a 3-structure with the special property that, over the target space, it measures the obstruction to lifting an  $E_8 \times \text{Spin}(n)$ -bundle to a certain nonabelian principal 2-bundle, called a *String 2-bundle*. Such a 3-structure is also known as a Chern-Simons 2-gerbe [21]. By itself this is abelian, but its structure is constrained by certain nonabelian data. Namely

this string 2-bundle with connection, from which the Chern-Simons 3-bundle arises, is itself an instance of a structure that yields parallel 2-transport. It can be described neither by abelian bundle gerbes, nor by Cheeger-Simons differential characters, nor by Deligne cohomology.

In anticipation of such situations, previous works have considered nonabelian gerbes and nonabelian bundle gerbes with connection. However, it turns out that care is needed in order to find the right setup. For instance, the kinds of nonabelian gerbes with connection studied in [17] [3], although very interesting, are not sufficiently general to capture String 2-bundles. Moreover, it is not easy to see how to obtain the parallel 2-transport assignment from these structures. For the application to string physics, it would be much more suitable to have a nonabelian generalization of the notion of a Cheeger-Simons differential character, and thus a structure which, by definition, knows how to assign generalized phases to  $n$ -dimensional paths.

The obvious generalization that is needed is that of a parallel transport  $n$ -functor. Such a notion was described in [9] [68]: a structure defined by the very fact that it labels  $n$ -paths by algebraic objects that allow composition in  $n$  different directions, such that this composition is compatible with the gluing of  $n$ -paths. One can show that such transport  $n$ -functors encompass abelian and nonabelian gerbes with connection as special cases [68]. However, these  $n$ -functors are more general. For instance, String 2-bundles with connection are given by parallel transport 2-functors. Ironically, the strength of the latter – namely their knowledge about general phase assignments to higher dimensional paths – is to some degree also a drawback: for many computations, a description *entirely* in terms of differential form data would be more tractable. However, the passage from parallel  $n$ -transport to the corresponding differential structure is more or less straightforward: a parallel transport  $n$ -functor is essentially a morphism of Lie  $n$ -groupoids. As such, it can be sent, by a procedure generalizing the passage from Lie groups to Lie algebras, to a morphism of Lie  $n$ -algebroids.

The aim of this paper is to describe two topics: First, to set up a formalism for higher bundles with connections entirely in terms of  $L_\infty$ -algebras, which may be thought of as a categorification of the theory of Cartan-Ehresmann connections. This is supposed to be the differential version of the theory of parallel transport  $n$ -functors, but an exhaustive discussion of the differentiation procedure is not given here. Instead we discuss a couple of examples and then show how the lifting problem has a nice description in this language. To do so, we present a family of  $L_\infty$ -algebras that govern the gauge structure of  $p$ -branes, as above, and discuss the lifting problem for them. By doing so, we characterize Chern-Simons 3-forms as local connection data on 3-bundles with connection which arise as the obstruction to lifts of ordinary bundles to the corresponding String 2-bundles, governed by the String Lie 2-algebra.

The formalism immediately allows the generalization of this situation to higher degrees. Indeed we indicate how certain 7-dimensional generalizations of Chern-Simons 3-bundles obstruct the lift of ordinary bundles to certain 6-bundles governed by the Fivebrane Lie 6-algebra. The latter correspond to what we define as the fivebrane structure, for which the degree seven NS field  $H_7$  plays the role that the degree three dual NS field  $H_3$  plays for the  $n = 2$  case.

The paper is organized in such a way that section 2 serves more or less as a self-contained description of the basic ideas and construction, with the rest of the document having all the details and all the proofs.

In this paper we make use of the homotopy algebras usually referred to as  $L_\infty$ -algebras. These algebras also go by other names such as sh-Lie algebras [57]. In our context we may also call such algebras Lie  $\infty$ -algebras which we think of as the abstract concept of an  $\infty$ -vector space with an antisymmetric bracket  $\infty$ -functor on it, which satisfies a Jacobi identity up to coherent equivalence, whereas “ $L_\infty$ -algebra” is concretely a codifferential coalgebra of sorts. In this paper we will nevertheless follow the standard notation of  $L_\infty$ -algebra.

## 2 The Setting and Plan

We set up a useful framework for describing higher order bundles with connection entirely in terms of Lie  $n$ -algebras, which can be thought of as arising from a categorification of the concept of an Ehresmann connection on a principal bundle. Then we apply this to the study of Chern-Simons  $n$ -bundles with connection as obstructions to lifts of principal  $G$ -bundles through higher String-like extensions of their structure Lie algebra.

### 2.1 $L_\infty$ -algebras and their String-like central extensions

A Lie group has all the right properties to locally describe the phase change of a charged particle as it traces out a worldline. A Lie  $n$ -group is a higher structure with precisely all the right properties to describe locally the phase change of a charged  $(n - 1)$ -brane as it traces out an  $n$ -dimensional worldvolume.

#### 2.1.1 $L_\infty$ -algebras

Just as ordinary Lie groups have Lie algebras, Lie  $n$ -groups have Lie  $n$ -algebras. If the Lie  $n$ -algebra is what is called *semistrict*, these are [5] precisely  $L_\infty$ -algebras [57] which have come to play a significant role in cohomological physics. A (“semistrict” and finite dimensional) Lie  $n$ -algebra is any of the following three equivalent structures:

- an  $L_\infty$ -algebra structure on a graded vector space  $\mathfrak{g}$  concentrated in the first  $n$  degrees  $(0, \dots, n - 1)$ ;
- a quasi-free differential graded-commutative algebra (“qDGCA”: free as a graded-commutative) algebra on the dual of that vector space: this is the Chevalley-Eilenberg algebra  $\text{CE}(\mathfrak{g})$  of  $\mathfrak{g}$ ;
- an  $n$ -category internal to the category of *graded* vector spaces and equipped with a skew-symmetric linear bracket functor which satisfies a Jacobi identity up to higher coherent equivalence.

For every  $L_\infty$ -algebra  $\mathfrak{g}$ , we have the following three qDGCA's:

- the **Chevalley-Eilenberg algebra**  $\text{CE}(\mathfrak{g})$
- the **Weil algebra**  $W(\mathfrak{g})$
- the algebra of **invariant polynomials** or **basic forms**  $\text{inv}(\mathfrak{g})$ .

These sit in a sequence

$$\text{CE}(\mathfrak{g}) \longleftarrow W(\mathfrak{g}) \longrightarrow \text{inv}(\mathfrak{g}) , \quad (1)$$

where all morphisms are morphisms of dg-algebras. This sequence plays the role of the sequence of differential forms on the “universal  $\mathfrak{g}$ -bundle”.

#### 2.1.2 $L_\infty$ -algebras from cocycles: String-like extensions

A simple but important source of examples for higher Lie  $n$ -algebras comes from the abelian Lie algebra  $\mathfrak{u}(1)$  which may be shifted into higher categorical degrees. We write  $b^{n-1}\mathfrak{u}(1)$  for the Lie  $n$ -algebra which is entirely trivial except in its  $n$ th degree, where it looks like  $\mathfrak{u}(1)$ . Just as  $\mathfrak{u}(1)$  corresponds to the Lie group  $U(1)$ , so  $b^{n-1}\mathfrak{u}(1)$  corresponds to the iterated classifying space  $B^{n-1}U(1)$ , realizable as the topological group given by the Eilenberg-MacLane space  $K(\mathbb{Z}, n)$ . Thus an important source for interesting Lie  $n$ -algebras comes from extensions

$$0 \rightarrow b^{n-1}\mathfrak{u}(1) \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0 \quad (2)$$

of an ordinary Lie algebra  $\mathfrak{g}$  by such a shifted abelian Lie  $n$ -algebra  $b^{n-1}\mathfrak{u}(1)$ . We find that, for each  $(n + 1)$ -cocycle  $\mu$  in the Lie algebra cohomology of  $\mathfrak{g}$ , we do obtain such a central extension, which we describe by

$$0 \rightarrow b^{n-1}\mathfrak{u}(1) \rightarrow \mathfrak{g}_\mu \rightarrow \mathfrak{g} \rightarrow 0 . \quad (3)$$

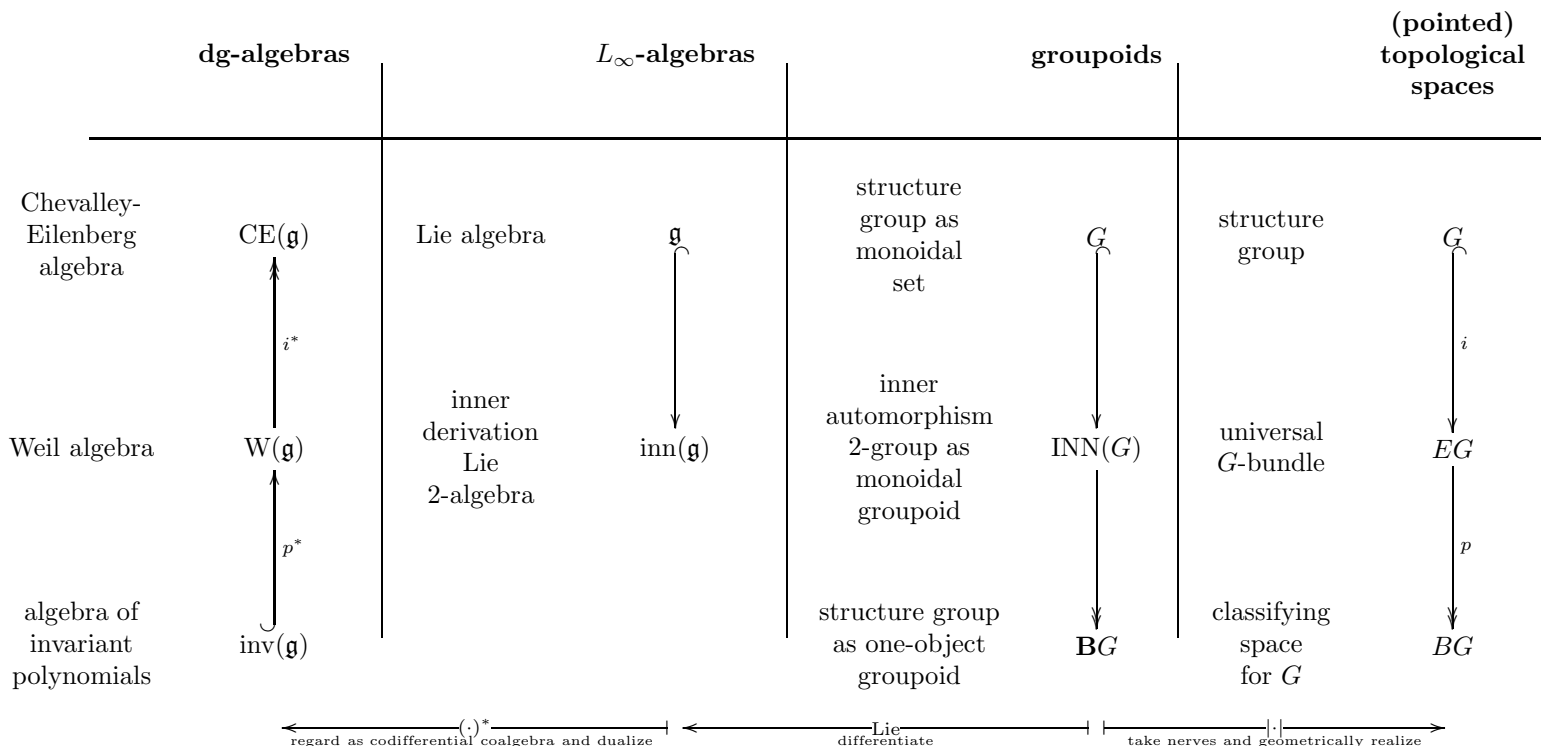


Figure 1: **The universal  $G$ -bundle** in its various incarnations. That the ordinary universal  $G$  bundle is the realization of the nerve of the groupoid which we denote here by  $\text{INN}(G)$  is an old result by Segal (see [65] for a review and a discussion of the situation for 2-bundles). This groupoid  $\text{INN}(G)$  is in fact a 2-group. The corresponding Lie 2-algebra (2-term  $L_\infty$ -algebra) we denote by  $\text{inn}(\mathfrak{g})$ . Regarding this as a codifferential coalgebra and then dualizing that to a differential algebra yields the Weil algebra of the Lie algebra  $\mathfrak{g}$ . This plays the role of differential forms on the universal  $G$ -bundle, as already known to Cartan. The entire table is expected to admit an  $\infty$ -ization. Here we concentrate on discussing  $\infty$ -bundles with connection in terms just of  $L_\infty$ -algebras and their dual dg-algebras. An integration of this back to the world of  $\infty$ -groupoids should proceed along the lines of [39, 43], but is not considered here.

Since, for the case when  $\mu = \langle \cdot, [\cdot, \cdot] \rangle$  is the canonical 3-cocycle on a semisimple Lie algebra  $\mathfrak{g}$ , this  $\mathfrak{g}_\mu$  is known ([7] and [43]) to be the Lie 2-algebra of the String 2-group, we call these central extensions *String-like* central extensions. (We also refer to these as Lie  $n$ -algebras “of Baez-Crans type” [5].) Moreover, whenever the cocycle  $\mu$  is related by transgression to an invariant polynomial  $P$  on the Lie algebra, we find that  $\mathfrak{g}_\mu$  fits into a short *homotopy* exact sequence of Lie  $(n+1)$ -algebras

$$0 \rightarrow \mathfrak{g}_\mu \rightarrow \mathrm{cs}_P(\mu) \rightarrow \mathrm{ch}_P(\mu) \rightarrow 0. \quad (4)$$

Here  $\text{cs}_P(\mathfrak{g})$  is a Lie  $(n+1)$ -algebra governed by the Chern-Simons term corresponding to the transgression element interpolating between  $\mu$  and  $P$ . In a similar fashion  $\text{ch}_P(\mathfrak{g})$  knows about the characteristic (Chern) class associated with  $P$ .

In summary, from elements of the cohomology of  $\text{CE}(\mathfrak{g})$  together with related elements in  $W(\mathfrak{g})$  we obtain the String-like extensions of Lie algebras to Lie  $2n$ -algebras and the associated Chern- and Chern-Simons Lie  $(2n - 1)$ -algebras:

Lie algebra cocycle	$\mu$	Baez-Crans Lie $n$ -algebra	$\mathfrak{g}_\mu$
invariant polynomial	$P$	Chern Lie $n$ -algebra	$\text{ch}_P(\mathfrak{g})$
transgression element	$\text{cs}$	Chern-Simons Lie $n$ -algebra	$\text{cs}_P(\mathfrak{g})$

### 2.1.3 $L_\infty$ -algebra differential forms

For  $\mathfrak{g}$  an ordinary Lie algebra and  $Y$  some manifold, one finds that dg-algebra morphisms  $\text{CE}(\mathfrak{g}) \rightarrow \Omega^\bullet(Y)$  from the Chevalley-Eilenberg algebra of  $\mathfrak{g}$  to the DGCA of differential forms on  $Y$  are in bijection with  $\mathfrak{g}$ -valued 1-forms  $A \in \Omega^1(Y, \mathfrak{g})$  whose ordinary curvature 2-form

$$F_A = dA + [A \wedge A] \quad (5)$$

vanishes. Without the flatness, the correspondence is with algebra morphisms *not* respecting the differentials. But dg-algebra morphisms  $A : W(\mathfrak{g}) \rightarrow \Omega^\bullet(Y)$  are in bijection with arbitrary  $\mathfrak{g}$ -valued 1-forms. These are flat precisely if  $A$  factors through  $\text{CE}(\mathfrak{g})$ . This situation is depicted in the following diagram:

$$\begin{array}{ccc} \text{CE}(\mathfrak{g}) & \xleftarrow{\quad} & W(\mathfrak{g}) \\ \downarrow (A, F_A=0) & & \downarrow (A, F_A) \\ \Omega^\bullet(Y) & \xrightarrow{\quad = \quad} & \Omega^\bullet(Y) \end{array} . \quad (6)$$

This has an obvious generalization for  $\mathfrak{g}$  an arbitrary  $L_\infty$ -algebra. For  $\mathfrak{g}$  any  $L_\infty$ -algebra, we write

$$\Omega^\bullet(Y, \mathfrak{g}) = \text{Hom}_{\text{dg-Alg}}(W(\mathfrak{g}), \Omega^\bullet(X)) \quad (7)$$

for the collection of  **$\mathfrak{g}$ -valued differential forms** and

$$\Omega_{\text{flat}}^\bullet(Y, \mathfrak{g}) = \text{Hom}_{\text{dg-Alg}}(\text{CE}(\mathfrak{g}), \Omega^\bullet(X)) \quad (8)$$

for the collection of **flat  $\mathfrak{g}$ -valued differential forms**.

## 2.2 $L_\infty$ -algebra Cartan-Ehresmann connections

### 2.2.1 $\mathfrak{g}$ -Bundle Descent data

A *descent object* for an ordinary principal  $G$ -bundle on  $X$  is a surjective submersion  $\pi : Y \rightarrow X$  together with a functor  $g : Y \times_X Y \rightarrow \mathbf{B}G$  from the groupoid whose morphisms are pairs of points in the same fiber of  $Y$ , to the groupoid  $\mathbf{B}G$  which is the one-object groupoid corresponding to the group  $G$ . Notice that the groupoid  $\mathbf{B}G$  is not itself the classifying space  $BG$  of  $G$ , but the geometric realization of its nerve,  $|\mathbf{B}G|$ , is:  $|\mathbf{B}G| = BG$ .

We may take  $Y$  to be the disjoint union of some open subsets  $\{U_i\}$  of  $X$  that form a good open cover of  $X$ . Then  $g$  is the familiar concept of a transition function describing a bundle that has been locally trivialized over the  $U_i$ . But one can also use more general surjective submersions. For instance, for  $P \rightarrow X$  any principal  $G$ -bundle, it is sometimes useful to take  $Y = P$ . In this case one obtains a canonical choice for the cocycle

$$g : Y \times_X Y = P \times_X P \rightarrow \mathbf{B}G \quad (9)$$

since  $P$  being principal means that

$$P \times_X P \simeq_{\text{diffeo}} P \times G. \quad (10)$$

This reflects the fact that every principal bundle canonically trivializes when pulled back to its own total space. The choice  $Y = P$  differs from that of a good cover crucially in the following aspect: if the group  $G$  is connected, then also the fibers of  $Y = P$  are connected. Cocycles over surjective submersions with connected fibers have special properties, which we will utilize: *When the fibers of  $Y$  are connected, we may*



think of the assignment of group elements to pairs of points in one fiber as arising from the parallel transport with respect to a flat vertical 1-form  $A_{\text{vert}} \in \Omega_{\text{vert}}^1(Y, \mathfrak{g})$ , flat along the fibers. As we shall see, this can be thought of as the vertical part of a Cartan-Ehresmann connection 1-form. This provides a morphism

$$\Omega_{\text{vert}}^\bullet(Y) \xleftarrow{A_{\text{vert}}} \text{CE}(\mathfrak{g}) \quad (11)$$

of differential graded algebras from the Chevalley-Eilenberg algebra of  $\mathfrak{g}$  to the vertical differential forms on  $Y$ .

Unless otherwise specified, *morphism* will always mean *homomorphism of differential graded algebra*.  $A_{\text{vert}}$  has an obvious generalization: for  $\mathfrak{g}$  any Lie  $n$ -algebra, we say that a  $\mathfrak{g}$ -bundle descent object for a  $\mathfrak{g}$ - $n$ -bundle on  $X$  is a surjective submersion  $\pi : Y \rightarrow X$  together with a morphism  $\Omega_{\text{vert}}^\bullet(Y) \xleftarrow{A_{\text{vert}}} \text{CE}(\mathfrak{g})$ . Now  $A_{\text{vert}} \in \Omega_{\text{vert}}^\bullet(Y, \mathfrak{g})$  encodes a collection of vertical  $p$ -forms on  $Y$ , each taking values in the degree  $p$ -part of  $\mathfrak{g}$  and all together satisfying a certain flatness condition, controlled by the nature of the differential on  $\text{CE}(\mathfrak{g})$ .

### 2.2.2 Connections on $n$ -bundles: the extension problem

Given a descent object  $\Omega_{\text{vert}}^\bullet(Y) \xleftarrow{A_{\text{vert}}} \text{CE}(\mathfrak{g})$  as above, a **flat connection** on it is an extension of the morphism  $A_{\text{vert}}$  to a morphism  $A_{\text{flat}}$  that factors through differential forms on  $Y$

$$\begin{array}{ccc} \Omega_{\text{vert}}^\bullet(Y) & \xleftarrow{A_{\text{vert}}} & \text{CE}(\mathfrak{g}) \\ \uparrow i^* & \swarrow A_{\text{flat}} & \\ \Omega^\bullet(Y) & & \end{array} \quad (12)$$

In general, such an extension does not exist. A general **connection** on a  $\mathfrak{g}$ -descent object  $A_{\text{vert}}$  is a morphism

$$\Omega^\bullet(Y) \xleftarrow{(A, F_A)} W(\mathfrak{g}) \quad (13)$$

from the Weil algebra of  $\mathfrak{g}$  to the differential forms on  $Y$  together with a morphism

$$\Omega^\bullet(Y) \xleftarrow{\{K_i\}} \text{inv}(\mathfrak{g}) \quad (14)$$

from the invariant polynomials on  $\mathfrak{g}$ , as in 2.1.1, to the differential forms on  $X$ , such that the following two squares commute:

$$\begin{array}{ccc} \Omega_{\text{vert}}^\bullet(Y) & \xleftarrow{A_{\text{vert}}} & \text{CE}(\mathfrak{g}) \\ \uparrow i^* & & \uparrow \\ \Omega^\bullet(Y) & \xleftarrow{(A, F_A)} & W(\mathfrak{g}) \\ \uparrow \pi^* & & \uparrow \\ \Omega^\bullet(X) & \xleftarrow{\{K_i\}} & \text{inv}(\mathfrak{g}) \end{array} \quad (15)$$

Whenever we have such two commuting squares, we say

- $A_{\text{vert}} \in \Omega_{\text{vert}}^\bullet(Y, \mathfrak{g})$  is a  $\mathfrak{g}$ -bundle **descent object** (playing the role of a **transition function**);
- $A \in \Omega^\bullet(Y, \mathfrak{g})$  is a (Cartan-Ehresmann) **connection** with values in the  $L_\infty$ -algebra  $\mathfrak{g}$  on the total space of the surjective submersion;
- $F_A \in \Omega^{\bullet+1}(Y, \mathfrak{g})$  are the corresponding **curvature** forms;
- and the set  $\{K_i \in \Omega^\bullet(X)\}$  are the corresponding **characteristic forms**, whose classes  $\{[K_i]\}$  in deRham cohomology

$$\Omega^\bullet(X) \xleftarrow{\{K_i\}} \text{inv}(\mathfrak{g}) \quad (16)$$

$$H_{\text{deRham}}^\bullet(X) \xleftarrow{\{[K_i]\}} H^\bullet(\text{inv}(\mathfrak{g}))$$

are the corresponding **characteristic classes** of the given descent object  $A_{\text{vert}}$ .

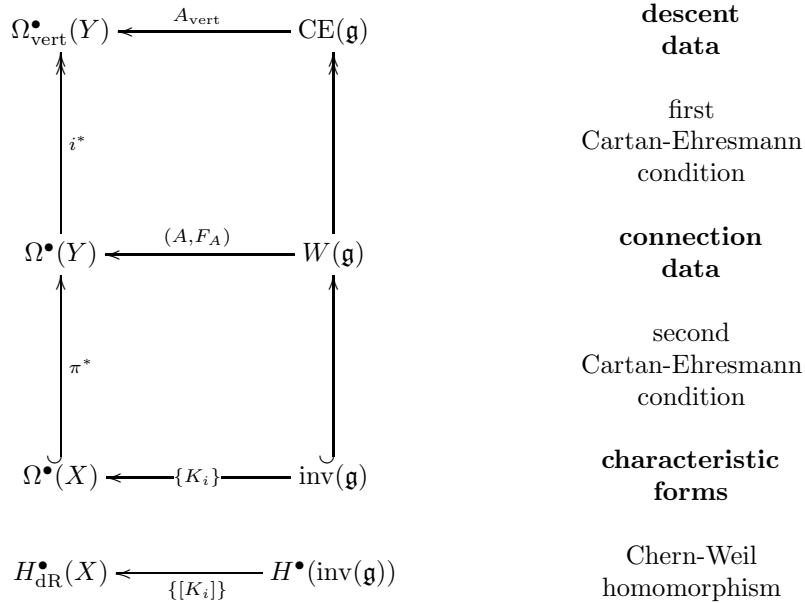


Figure 2: **A  $\mathfrak{g}$ -connection descent object and its interpretation.** For  $\mathfrak{g}$ -any  $L_\infty$ -algebra and  $X$  a smooth space, a  $\mathfrak{g}$ -connection on  $X$  is an equivalence class of pairs  $(Y, (A, F_A))$  consisting of a surjective submersion  $\pi : Y \rightarrow X$  and dg-algebra morphisms forming the above commuting diagram. The equivalence relation is concordance of such diagrams. The situation for ordinary Cartan-Ehresmann (1-)connections is described in 7.2.1.

Hence we realize the curvature of a  $\mathfrak{g}$ -connection as the *obstruction* to extending a  $\mathfrak{g}$ -descent object to a *flat*  $\mathfrak{g}$ -connection.

### 2.3 Higher String and Chern-Simons $n$ -transport: the lifting problem

Given a  $\mathfrak{g}$ -descent object

$$\begin{array}{ccc} & & \text{CE}(\mathfrak{g}) , \\ & \swarrow A_{\text{vert}} & \\ \Omega_{\text{vert}}^{\bullet}(Y) & & \end{array} \quad (17)$$

and given an extension of  $\mathfrak{g}$  by a String-like  $L_{\infty}$ -algebra

$$\text{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{i} \text{CE}(\mathfrak{g}_{\mu}) \xleftarrow{\quad} \text{CE}(\mathfrak{g}) , \quad (18)$$

we ask if it is possible to *lift the descent object* through this extension, i.e. to find a dotted arrow in

$$\begin{array}{ccc} \text{CE}(b^{n-1}\mathfrak{u}(1)) & \xleftarrow{\quad} & \text{CE}(\mathfrak{g}_{\mu}) \xleftarrow{\quad} \text{CE}(\mathfrak{g}) . \\ & \searrow \text{dotted} & \swarrow A_{\text{vert}} \\ & & \Omega_{\text{vert}}^{\bullet}(Y) \end{array} \quad (19)$$

In general this is not possible. We seek a straightforward way to compute the obstruction to the existence of the lift. The strategy is to form the *weak* (homotopy) kernel of

$$\text{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{i} \text{CE}(\mathfrak{g}_{\mu}) \quad (20)$$

which we denote by  $\text{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_{\mu})$  and realize as a mapping cone of qDGCAs. This comes canonically with a morphism  $f$  from  $\text{CE}(\mathfrak{g})$  which happens to have a *weak* inverse

$$\begin{array}{ccc} & & \text{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_{\mu}) . \\ & \nearrow f & \nwarrow f^{-1} \\ \text{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{i} \text{CE}(\mathfrak{g}_{\mu}) & \xleftarrow{\quad} & \text{CE}(\mathfrak{g}) \end{array} \quad (21)$$

Then we see that, while the lift to a  $\mathfrak{g}_{\mu}$ -cocycle may not always exist, the lift to a  $(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_{\mu})$ -cocycle does always exist. We form  $A_{\text{vert}} \circ f^{-1}$ :

$$\begin{array}{ccc} & & \text{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_{\mu}) . \\ & \nearrow f^{-1} & \\ \text{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{i} \text{CE}(\mathfrak{g}_{\mu}) & \xleftarrow{\quad} & \text{CE}(\mathfrak{g}) \\ & \searrow A_{\text{vert}} & \swarrow \\ & & \Omega_{\text{vert}}^{\bullet}(Y) \end{array} \quad (22)$$

The failure of this lift to be a true lift to  $\mathfrak{g}_{\mu}$  is measured by the component of  $A_{\text{vert}} \circ f^{-1}$  on  $b^{n-1}\mathfrak{u}(1)[1] \simeq b^n\mathfrak{u}(1)$ . Formally this is the composite  $A'_{\text{vert}} := A_{\text{vert}} \circ f^{-1} \circ j$  in

$$\begin{array}{ccc} & & \text{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_{\mu}) \xleftarrow{j} \text{CE}(b^n\mathfrak{u}(1)) . \\ & \nearrow f^{-1} & \nwarrow A'_{\text{vert}} \\ \text{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{i} \text{CE}(\mathfrak{g}_{\mu}) & \xleftarrow{\quad} & \text{CE}(\mathfrak{g}) \\ & \searrow A_{\text{vert}} & \swarrow \\ & & \Omega_{\text{vert}}^{\bullet}(Y) \end{array} \quad (23)$$

The nontriviality of the  $b^n\mathfrak{u}(1)$ -descent object  $A'_{\text{vert}}$  is the obstruction to constructing the desired lift.

We thus find the following results, for any  $\mathfrak{g}$ -cocycle  $\mu$  which is in transgression with the invariant polynomial  $P$  on  $\mathfrak{g}$ ,

- The characteristic classes (in deRham cohomology) of  $\mathfrak{g}_\mu$ -bundles are those of the corresponding  $\mathfrak{g}$ -bundles modulo those coming from the invariant polynomial  $P$ .
- The lift of a  $\mathfrak{g}$ -valued connection to a  $\mathfrak{g}_\mu$ -valued connection is obstructed by a  $b^n\mathfrak{u}(1)$ -valued  $(n+1)$ -connection whose  $(n+1)$ -form curvature is  $P(F_A)$ , i.e. the image under the Chern-Weil homomorphism of the invariant polynomial corresponding to  $\mu$ .
- Accordingly, the  $(n+1)$ -form connection of the obstructing  $b^n\mathfrak{u}(1)$   $(n+1)$ -bundle is a Chern-Simons form for this characteristic class.

We call the obstructing  $b^n\mathfrak{u}(1)$   $(n+1)$ -descent object the corresponding Chern-Simons  $(n+1)$ -bundle. For the case when  $\mu = \langle \cdot, [\cdot, \cdot] \rangle$  is the canonical 3-cocycle on a semisimple Lie algebra  $\mathfrak{g}$ , this structure (corresponding to a 2-gerbe) has a 3-connection given by the ordinary Chern-Simons 3-form and has a curvature 4-form given by the (image in deRham cohomology of) the first Pontrjagin class of the underlying  $\mathfrak{g}$ -bundle.

### 3 Physical applications: String-, Fivebrane- and $p$ -Brane structures

We can now discuss physical applications of the formalism that we have developed. What we describe is a useful way to handle obstructing  $n$ -bundles of various kinds that appear in string theory. In particular, we can describe generalizations of string structure in string theory. In the context of  $p$ -branes, such generalizations have been suggested based on  $p$ -loop spaces [30] [10] [64] and, more generally, on the space of maps  $\text{Map}(M, X)$  from the brane worldvolume  $M$  to spacetime  $X$  [60]. The statements in this section will be established in detail in [77].

From the point of view of supergravity, all branes, called  $p$ -branes in that setting, are a priori treated in a unified way. In tracing back to string theory, however, there is a distinction in the form-fields between the Ramond-Ramond (RR) and the Neveu-Schwarz (NS) forms. The former live in generalized cohomology and the latter play two roles: they act as twist fields for the RR fields and they are also connected to the geometry and topology of spacetime. The  $H$ -field  $H_3$  plays the role of a twist in K-theory for the RR fields [50] [15] [59]. The twist for the degree seven dual field  $H_7$  is observed in [67] at the rational level.

The ability to define fields and their corresponding partition functions puts constraints on the topology of the underlying spacetime. The most commonly understood example is that of fermions where the ability to define them requires spacetime to be spin, and the ability to describe theories with chiral fermions requires certain restrictions coming from the index theorem. In the context of heterotic string theory, the Green-Schwarz anomaly cancelation leads to the condition that the difference between the Pontrjagin classes of the tangent bundle and that of the gauge bundle be zero. This is called the string structure, which can be thought of as a spin structure on the loop space of spacetime [52] [27]. In M-theory, the ability to define the partition function leads to an anomaly given by the integral seventh-integral Steifel-Whitney class of spacetime [29] whose cancelation requires spacetime to be orientable with respect to generalized cohomology theories beyond K-theory [55].

In all cases, the corresponding structure is related to the homotopy groups of the orthogonal group: the spin structure amounts to killing the first homotopy group, the string structure and – to some extent – the  $W_7$  condition to killing the third homotopy group. Note that when we say that the  $n$ -th homotopy group is killed, we really mean that all homotopy groups up to and including the  $n$ -th one are killed. For instance,

a String structure requires killing everything up to and including the third, hence everything through the sixth, since there are no homotopy groups in degrees four, five or six.

The Green-Schwarz anomaly cancelation condition for the heterotic string can be translated to the language of  $n$ -bundles as follows. We have two bundles, the spin bundle with structure group  $G = \text{Spin}(10)$ , and the gauge bundle with structure group  $G'$  being either  $\text{SO}(32)/\mathbb{Z}_2$  or  $E_8 \times E_8$ . Considering the latter, we have one copy of  $E_8$  on each ten-dimensional boundary component, which can be viewed as an end-of-the-world nine-brane, or M9-brane [47]. The structure of the four-form on the boundary which we write as

$$G_4|_{\partial} = dH_3 \quad (24)$$

implies that the 3-bundle (2-gerbe) becomes the trivializable lifting 2-gerbe of a  $\text{String}(\text{Spin}(10) \times E_8)$  bundle over the M9-brane. As the four-form contains the difference of the Pontrjagin classes of the bundles with structure groups  $G$  and  $G'$ , the corresponding three-form will be a difference of Chern-Simons forms. The bundle aspect of this has been studied in [12] and will be revisited in the current context in [77].

The NS fields play a special role in relation to the homotopy groups of the orthogonal group. The degree three class  $[H_3]$  plays the role of a twist for a spin structure. Likewise, the degree seven class plays a role of a twist for a higher structure related to  $BO\langle 9 \rangle$ , the 8-connected cover of  $BO$ , which we might call a *Fivebrane*-structure on spacetime. We can talk about such a structure once the spacetime already has a string structure. The obstructions are given in table 2, where  $A$  is the connection on the  $G'$  bundle and  $\omega$  is a connection on the  $G$  bundle.

$n$	2 $= 4 \cdot 0 + 2$	6 $= 4 \cdot 1 + 2$
fundamental object		
$(n - 1)$ -brane	string	5-brane
$n$ -particle		
target space structure	string structure $\text{ch}_2(A) - p_1(\omega) = 0$	fivebrane structure $\text{ch}_4(A) - \frac{1}{48}p_2(\omega) = 0$

Table 2: **Higher dimensional extended objects** and the corresponding topological structures.

In the above we alluded to how the brane structures are related to obstructions to having spacetimes with connected covers of the orthogonal groups as structures. The obstructing classes here may be regarded as classifying the corresponding obstructing  $n$ -bundles, after we apply the general formalism that we outlined earlier. The main example of this general mechanism that will be of interest to us here is the case where  $\mathfrak{g}$  is an ordinary semisimple Lie algebra. In particular, we consider  $\mathfrak{g} = \mathfrak{spin}(n)$ . For  $\mathfrak{g} = \mathfrak{spin}(n)$  and  $\mu$  a  $(2n + 1)$ -cocycle on  $\mathfrak{spin}(n)$ , we call  $\mathfrak{spin}(n)_\mu$  the (skeletal version of the)  $(2n - 1)$ -**brane Lie**  $(2n) - \text{algebra}$ . Thus, the case of String structure and Fivebrane structure occurring in the fundamental string and NS fivebrane correspond to the cases  $n = 1$  and  $n = 3$  respectively. Now applying our formalism for  $\mathfrak{g} = \mathfrak{spin}(n)$ , and  $\mu_3, \mu_7$  the canonical 3- and 7-cocycle, respectively, we have:

- the obstruction to lifting a  $\mathfrak{g}$ -bundle descent object to a String 2-bundle (a  $\mathfrak{g}_{\mu_3}$ -bundle descent object) is the first Pontryagin class of the original  $\mathfrak{g}$ -bundle cocycle;
- the obstruction to lifting a String 2-bundle descent object to a Fivebrane 6-bundle cocycle (a  $\mathfrak{g}_{\mu_7}$ -bundle descent object) is the second Pontryagin class of the original  $\mathfrak{g}$ -bundle cocycle.

The cocycles and invariant polynomials corresponding to the two structures are given in the following table

In case of the fundamental string, the obstruction to lifting the  $PU(H)$  bundles to  $U(H)$  bundles is measured by a gerbe or a line 2-bundle. In the language of  $E_8$  bundles this corresponds to lifting the loop

<i>p</i> -brane	cocycle	invariant polynomial	
$p = 1 = 4 \cdot 0 + 1$	$\mu_3 = \langle \cdot, [\cdot, \cdot] \rangle$	$P_1 = \langle \cdot, \cdot \rangle$	first Pontrjagin
$p = 5 = 4 \cdot 1 + 1$	$\mu_7 = \langle \cdot, [\cdot, \cdot], [\cdot, \cdot], [\cdot, \cdot] \rangle$	$P_2 = \langle \cdot, \cdot, \cdot, \cdot \rangle$	second Pontrjagin

Table 3: **Lie algebra cohomology governing NS  $p$ -branes.**

group  $LE_8$  bundles to the central extension  $\hat{LE}_8$  bundles [59]. The obstruction for the case of the String structure is a 2-gerbe and that of a Fivebrane structure is a 6-gerbe. The structures are summarized in the following table

obstruction		$G$ -bundle		$\hat{G}$ -bundle
1-gerbes / line 2-bundles		$PU(H)$ -bundles		$U(H)$ -bundles
2-gerbes / line 3-bundles	obstruct the lift of	$Spin(n)$ -bundles	to	$String(n)$ -2-bundles
6-gerbes / line 7-bundles		$String(n)$ -2-bundles		FiveBrane( $n$ )-6-bundles

Table 4: **Obstructing line  $n$ -bundles** appearing in string theory.

A description can also be given in terms of (higher) loop spaces, generalizing the known case where a String structure on a space  $X$  can be viewed as a Spin structure on the loop space  $LX$ . A fuller discussion of the ideas of this section will be given in [77].

## 4 Statement of the main results

We define, for any  $L_\infty$ -algebra  $\mathfrak{g}$  and any smooth space  $X$ , a notion of

- $\mathfrak{g}$ -descent objects over  $X$ ;

and an extension of these to

- $\mathfrak{g}$ -connection descent objects over  $X$ .

These descent objects are to be thought of as the data obtained from locally trivializing an  $n$ -bundle (with connection) whose structure  $n$ -group has the Lie  $n$ -algebra  $\mathfrak{g}$ . Being differential versions of  $n$ -functorial descent data of such  $n$ -bundles, they consist of morphisms of quasi free differential graded-commutative algebras (qDGCAs).

We define for each  $L_\infty$ -algebra  $\mathfrak{g}$  a dg-algebra  $\text{inv}(\mathfrak{g})$  of *invariant polynomials* on  $\mathfrak{g}$ . We show that every  $\mathfrak{g}$ -connection descent object gives rise to a collection of deRham classes on  $X$ : its *characteristic classes*. These are images of the elements of  $\text{inv}(\mathfrak{g})$ . Two descent objects are taken to be equivalent if they are *concordant* in a natural sense.

Our first main result is

**Theorem 1 (characteristic classes)** *Characteristic classes are indeed characteristic of  $\mathfrak{g}$ -descent objects (but do not necessarily fully characterize them) in the following sense:*

- *Concordant  $\mathfrak{g}$ -connection descent objects have the same characteristic classes.*
- *If the  $\mathfrak{g}$ -connection descent objects differ just by a gauge transformation, they even have the same characteristic forms.*

This is our proposition 32 and corollary 2.

**Remark.** We expect that this result can be strengthened. Currently our characteristic classes are just in deRham cohomology. One would expect that these are images of classes in integral cohomology. While we do not attempt here to discuss integral characteristic classes in general, we discuss some aspects of this for the case of abelian Lie  $n$ -algebras  $\mathfrak{g} = b^{n-1}\mathfrak{u}(1)$  in 7.1.1 by relating  $\mathfrak{g}$ -descent objects to Deligne cohomology.

The reader should also note that in our main examples to be discussed in section 8 we start with an  $L_\infty$ -connection which happens to be an ordinary Cartan-Ehresmann connection on an ordinary bundle and is hence known to have integral classes. It follows from our results then that also the corresponding Chern-Simons 3-connections in particular have an integral class.

We define String-like extensions  $\mathfrak{g}_\mu$  of  $L_\infty$ -algebras coming from any  $L_\infty$ -algebra cocycle  $\mu$ : a closed element in the Chevalley-Eilenberg dg-algebra  $\text{CE}(\mathfrak{g})$  corresponding to  $\mathfrak{g}$ :  $\mu \in \text{CE}(\mathfrak{g})$ . These generalize the String Lie 2-algebra which governs the dynamics of (heterotic) superstrings.

Our second main results is

**Theorem 2 (string-like extensions and their properties)** *Every degree  $(n+1)$ -cocycle  $\mu$  on an  $L_\infty$ -algebra  $\mathfrak{g}$  we obtain the string-like extension  $\mathfrak{g}_\mu$  which sits in an exact sequence*

$$0 \rightarrow b^{n-1}\mathfrak{u}(1) \rightarrow \mathfrak{g}_\mu \rightarrow \mathfrak{g} \rightarrow 0.$$

*When  $\mu$  is in transgression with an invariant polynomial  $P$  we furthermore obtain a weakly exact sequence*

$$0 \rightarrow \mathfrak{g}_\mu \rightarrow \text{cs}_P(\mathfrak{g}) \rightarrow \text{ch}_P(\mu) \rightarrow 0$$

*of  $L_\infty$ -algebras, where  $\text{cs}_P(\mathfrak{g}) \simeq \text{inn}(\mathfrak{g}_\mu)$  is trivializable (equivalent to the trivial  $L_\infty$ -algebra). There is an algebra of invariant polynomials on  $\mathfrak{g}$  associated with  $\text{cs}_P(\mathfrak{g})$  and we show that it is the algebra of invariant polynomials of  $\mathfrak{g}$  modulo the ideal generated by  $P$ .*

This is proposition 20, proposition 21 and proposition 24.

Our third main result is

**Theorem 3 (obstructions to lifts through String-like extensions)** *For  $\mu \in \text{CE}(\mathfrak{g})$  any degree  $n+1$   $\mathfrak{g}$ -cocycle that transgresses to an invariant polynomial  $P \in \text{inv}(\mathfrak{g})$ , the obstruction to lifting a  $\mathfrak{g}$ -descent object to a  $\mathfrak{g}_\mu$ -descent object is a  $(b^n\mathfrak{u}(1))$ -descent object whose single characteristic class is the class corresponding to  $P$  of the original  $\mathfrak{g}$ -descent object.*

*This is reflected by the fact that the cohomology of the basic forms on the Chevalley-Eilenberg algebra of the corresponding Chern-Simons  $L_\infty$ -algebra  $\text{cs}_P(\mathfrak{g})$  is that of the algebra of basic forms on  $\text{inv}(\mathfrak{g})$  modulo the ideal generated by  $P$ .*

This is our proposition 24 and proposition 44.

We discuss the following **applications**.

- For  $\mathfrak{g}$  an ordinary semisimple Lie algebra and  $\mu$  its canonical 3-cocycle, the obstruction to lifting a  $\mathfrak{g}$ -bundle to a String 2-bundle is a Chern-Simons 3-bundle with characteristic class the Pontrjagin class of the original bundle. This is a special case of our proposition 44 which is spelled out in detail in 8.3.1.

The vanishing of this obstruction is known as a String structure [52, 56, 63]. In categorical language, this issue was first discussed in [72].

By passing from our Lie  $\infty$ -algebraic description to smooth spaces along the lines of 5.1 and then forming fundamental  $n$ -groupoids of these spaces, one can see that our construction of obstructing  $n$ -bundles to lifts through String-like extensions reproduces the construction [18, 19] of Čech cocycles representing characteristic classes. This, however, will not be discussed here.

- This result generalizes to all String-like extensions. Using the 7-cocycle on  $\mathfrak{so}(n)$  we obtain lifts through extensions by a Lie 6-algebra, which we call the Fivebrane Lie 6-algebra. Accordingly, fivebrane structures on string structures are obstructed by the second Pontrjagin class.

This pattern continues and one would expect our obstruction theory for lifts through string-like extensions with respect to the 11-cocycle on  $\mathfrak{so}(n)$  to correspond to *Ninebrane* structure.

The issue of  $p$ -brane structures for higher  $p$  was discussed before in [60]. In contrast to the discussion there, we here see  $p$ -brane structures only for  $p = 4n + 1$ , corresponding to the list of invariant polynomials and cocycles for  $\mathfrak{so}(n)$ . While our entire obstruction theory applies to all cocycles on all Lie  $\infty$ -algebras, it is only for those on  $\mathfrak{so}(n)$  and maybe  $\mathfrak{e}_8$  for which the physical interpretation in the sense of  $p$ -brane structures is understood.

- We discuss how the action functional of the topological field theory known as BF-theory arises from an invariant polynomial on a strict Lie 2-algebra, in a generalization of the integrated Pontrjagin 4-form of the topological term in Yang-Mills theory. See proposition 18 and the example in 6.6.1.

This is similar to but different from the Lie 2-algebraic interpretation of BF theory indicated in [37, 38], where the “cosmological” bilinear in the connection 2-form is not considered and a constraint on the admissible strict Lie 2-algebras is imposed.

- We indicate in 9.1 the notion of parallel transport induced by a  $\mathfrak{g}$ -connection, relate it to the  $n$ -functorial parallel transport of [9, 68, 69, 70] and point out how this leads to  $\sigma$ -model actions in terms of dg-algebra morphisms. See section 9.
- We indicate in 9.3.1 how by forming configuration spaces by sending DGCA's to smooth spaces and then using the internal hom of smooth space, we obtain for every  $\mathfrak{g}$ -connection descent object configuration spaces of maps equipped with an action functional induced by the transgressed  $\mathfrak{g}$ -connection. We show that the algebra of differential forms on these configuration spaces naturally supports the structure of the corresponding BRST-BV complex, with the iterated ghost-of-ghost structure inherited from the higher degree symmetries induced by  $\mathfrak{g}$ .

This construction is similar in spirit to the one given in [1], reviewed in [66], but also, at least superficially, a bit different.

- We indicate also in 9.3.1 how this construction of configuration spaces induces the notion of transgression of  $n$ -bundles on  $X$  to  $(n - k)$ -bundles on spaces of maps from  $k$ -dimensional spaces into  $X$ . An analogous integrated description of transgression in terms of inner homs is in [69]. We show in 9.3.1 in particular that this transgression process relates the concept of String-structures in terms of 4-classes down on  $X$  with the corresponding 3-classes on  $LX$ , as discussed for instance in [56]. Our construction immediately generalizes to fivebrane and higher classes.

All of our discussion here pertains to *principal*  $L_\infty$ -connections. One can also discuss *associated*  $\mathfrak{g}$ -connections induced by  $(\infty)$ -representations of  $\mathfrak{g}$  (for instance as in [58]) and then study the collections of “sections” or “modules” of such associated  $\mathfrak{g}$ -connections.

The extended quantum field theory of a  $(n - 1)$ -brane charged under an  $n$ -connection (“a charged  $n$ -particle”, definition 42) should (see for instance [32, 33, 72, 45]) assign to each  $d$ -dimensional part  $\Sigma$  of the brane’s parameter space (“worldvolume”) the collection (an  $(n - d - 1)$ -category, really) of sections/modules of the transgression of the  $n$ -bundle to the configuration space of maps from  $\Sigma$ .

For instance, the space of sections of a Chern-Simons 3-connection transgressed to maps from the circle should yield the representation category of the Kac-Moody extension of the corresponding loop group.

Our last proposition 47 points in this direction. But a more detailed discussion will not be given here.



## 5 Differential graded-commutative algebra

Differential  $\mathbb{N}$ -graded commutative algebras (DGCAs) play a prominent role in our discussion. One way to understand what is special about DGCAs is to realize that every DGCA can be regarded, essentially, as the algebra of *differential forms on some generalized smooth space*.

We explain what this means precisely in 5.1. The underlying phenomenon is essentially the familiar governing principle of Sullivan models in rational homotopy theory [44, 73], but instead of working with simplicial spaces, we here consider presheaf categories. This will not become relevant, though, until the discussion of configuration spaces, parallel transport and action functionals in 9.

### 5.1 Differential forms on smooth spaces

We can think of every differential graded commutative algebra essentially as being the algebra of differential forms on *some* space, possibly a generalized space.

**Definition 1** *Let  $S$  be the category whose objects are the open subsets of  $\mathbb{R} \cup \mathbb{R}^2 \cup \mathbb{R}^3 \cup \dots$  and whose morphisms are smooth maps between these. We write*

$$S^\infty := \text{Set}^{S^{\text{op}}} \quad (25)$$

*for the category of set-valued presheaves on  $S$ .*

So an object  $X$  in  $S^\infty$  is an assignment of sets  $U \mapsto X(U)$  to each open subset  $U$ , together with an assignment

$$(U \xrightarrow{\phi} V) \mapsto (X(U) \xleftarrow{\phi_X^*} X(V)) \quad (26)$$

of maps of sets to maps of smooth subsets which respects composition. A morphism

$$f : X \rightarrow Y \quad (27)$$

of smooth spaces is an assignment  $U \mapsto (X(U) \xrightarrow{f_U} Y(U))$  of maps of sets to open subsets, such that for

all smooth maps of subsets  $U \xrightarrow{\phi} V$  we have that the square

$$\begin{array}{ccc} X(V) & \xrightarrow{f_V} & Y(V) \\ \downarrow \phi_X^* & & \downarrow \phi_Y^* \\ X(U) & \xrightarrow{f_U} & Y(U) \end{array} \quad (28)$$

commutes. We think of the objects of  $S^\infty$  smooth spaces. The set  $X(U)$  that such a smooth space  $X$  assigns to an open subset  $U$  is to be thought of as the set of smooth maps from  $U$  into  $X$ . As opposed to manifolds which are *locally isomorphic* to an object in  $S$ , smooth spaces can hence be thought of as being objects which are just required to have the property that they may be *probed* by objects of  $S$ . Every open subset  $V$  becomes a smooth space by setting

$$V : U \mapsto \text{Hom}_{S^\infty}(U, V). \quad (29)$$

This are the *representable* presheaves. Similarly, every ordinary manifold  $X$  becomes a smooth space by setting

$$X : U \mapsto \text{Hom}_{\text{manifolds}}(U, X). \quad (30)$$

The special property of smooth spaces which we need here is that they form a (cartesian) *closed* category:

- for any two smooth spaces  $X$  and  $Y$  there is a cartesian product  $X \times Y$ , which is again a smooth space, given by the assignment

$$X \times Y : U \mapsto X(U) \times Y(U); \quad (31)$$

where the cartesian product on the right is that of sets;

- the collection  $\text{hom}(X, Y)$  of morphisms from one smooth space  $X$  to another smooth space  $Y$  is again a smooth space, given by the assignment

$$\text{hom}_{S^\infty}(X, Y) : U \mapsto \text{Hom}_{S^\infty}(X \times U, Y). \quad (32)$$

A very special smooth space is the smooth space of differential forms.

**Definition 2** We write  $\Omega^\bullet$  for the smooth space which assigns to each open subset the set of differential forms on it

$$\Omega^\bullet : U \mapsto \Omega^\bullet(U). \quad (33)$$

Using this object we define the DGCA of differential forms on any smooth space  $X$  to be the set

$$\Omega^\bullet(X) := \text{Hom}_{S^\infty}(X, \Omega^\bullet) \quad (34)$$

equipped with the obvious DGCA structure induced by the local DGCA structure of each  $\Omega^\bullet(U)$ .

Therefore the object  $\Omega^\bullet$  is in a way both a smooth space as well as a differential graded commutative algebra: it is a DGCA-valued presheaf. Such objects are known as *schizophrenic* [48] or better *ambimorphic* [78] objects: they relate two different worlds by duality. In fact, the process of mapping into these objects provides an adjunction between the dual categories:

**Definition 3** There are contravariant functors from smooth spaces to DGCAs given by

$$\begin{aligned} \Omega^\bullet : S^\infty &\rightarrow \text{DGCAs} \\ X &\mapsto \Omega^\bullet(X) \end{aligned} \quad (35)$$

and

$$\begin{aligned} \text{Hom}(-, \Omega^\bullet(-)) : \text{DGCAs} &\rightarrow S^\infty \\ A &\mapsto X_A \end{aligned} \quad (36)$$

These form an adjunction of categories. The unit

$$\begin{array}{ccccc} & & \text{Id} & & \\ & \swarrow & \Downarrow^u & \searrow & \\ \text{DGCAs} & \xrightarrow{\text{Hom}(-, \Omega^\bullet(-))} & S^\infty & \xrightarrow{\Omega^\bullet} & \text{DGCAs} \end{array} \quad (37)$$

of this adjunction is a natural transformation whose component map embeds each DGCA  $A$  into the algebra of differential forms on the smooth space it defines

$$A \hookrightarrow \Omega^\bullet(X_A) \quad (38)$$

by sending every  $a \in A$  to the map of presheaves

$$(f \in \text{Hom}_{\text{DGCAs}}(A, \Omega^\bullet(U))) \mapsto (f(a) \in \Omega^\bullet(U)). \quad (39)$$

This way of obtaining forms on  $X_A$  from elements of  $A$  will be crucial for our construction of differential forms on spaces of maps,  $\text{hom}(X, Y)$ , used in 9.3.

Using this adjunction, we can “pull back” the internal hom of  $S^\infty$  to DGCAs. Since the result is not literally the internal hom in DGCAs (which does not exist since DGCAs are not *profinite* as opposed to codifferential coalgebras [39]) we call it “maps” instead of “hom”.

**Definition 4 (forms on spaces of maps)** *Given any two DGCA's  $A$  and  $B$ , we define the DGCA of "maps" from  $B$  to  $A$*

$$\text{maps}(B, A) := \Omega^\bullet(\text{hom}_{S^\infty}(X_A, X_B)). \quad (40)$$

This is a functor

$$\text{maps} : \text{DGCA's}^{\text{op}} \times \text{DGCA's} \rightarrow \text{DGCA's}. \quad (41)$$

Notice the fact (for instance corollary 35.10 in [53] and theorem 2.8 in [61]) that for any two smooth spaces  $X$  and  $Y$ , algebra homomorphisms  $C^\infty(X) \xleftarrow{\phi^*} C^\infty(Y)$  and hence DGCA morphisms  $\Omega^\bullet(X) \xleftarrow{\phi^*} \Omega^\bullet(Y)$  are in bijection with smooth maps  $\phi : X \rightarrow Y$ .

It follows that an element of  $\text{hom}(X_A, X_B)$  is, over test domains  $U$  and  $V$  a natural map of sets

$$\text{Hom}_{\text{DGCA's}}(A, \Omega^\bullet(V)) \times \text{Hom}_{\text{DGCA's}}(\Omega^\bullet(U), \Omega^\bullet(V)) \rightarrow \text{Hom}_{\text{DGCA's}}(B, \Omega^\bullet(V)). \quad (42)$$

One way to obtain such maps is from pullback along algebra homomorphisms

$$B \rightarrow A \otimes \Omega^\bullet(U).$$

This will be an important source of DGCA's of maps for the case that  $A$  is the Chevalley-Eilenberg algebra of an  $L_\infty$ -algebra, as described in 5.1.1.

### 5.1.1 Examples

**Diffeological spaces** Particularly useful are smooth spaces  $X$  which, while not quite manifolds, have the property that there is a set  $X_s$  such that

$$X : U \mapsto X(U) \subset \text{Hom}_{\text{Set}}(U, X_s) \quad (43)$$

for all  $U \in S$ . These are the Chen-smooth or diffeological spaces used in [9, 69, 70]. In particular, all spaces of maps  $\text{hom}_{S^\infty}(X, Y)$  for  $X$  and  $Y$  manifolds are of this form. This includes in particular loop spaces.

**Forms on spaces of maps.** When we discuss parallel transport and its transgression to configuration spaces in 9.3, we need the following construction of differential forms on spaces of maps.

**Definition 5 (currents)** *For  $A$  any DGCA, we say that a current on  $A$  is a smooth linear map*

$$c : A \rightarrow \mathbb{R}. \quad (44)$$

For  $A = \Omega^\bullet(X)$  this reduces to the ordinary notion of currents.

**Proposition 1** *Let  $A$  be a quasi free DGCA's in positive degree (meaning that the underlying graded commutative algebras are freely generated from some graded vector space in positive degree). For each element  $b \in B$  and current  $c$  on  $A$ , we get an element in  $\Omega^\bullet(\text{Hom}_{\text{DGCA's}}(B, A \otimes \Omega^\bullet(-)))$  by mapping, for each  $U \in S$*

$$\begin{aligned} \text{Hom}_{\text{DGCA's}}(B, A \otimes \Omega^\bullet(U)) &\rightarrow \Omega^\bullet(U) \\ f^* &\mapsto c(f^*(b)). \end{aligned} \quad (45)$$

If  $b$  is in degree  $n$  and  $c$  in degree  $m \leq n$ , then this differential form is in degree  $n - m$ .

**The superpoint.** Most of the DGCA's we shall consider here are non-negatively graded or even positively graded. These can be thought of as Chevalley-Eilenberg algebras of Lie  $n$ -algebroids and Lie  $n$ -algebras, respectively, as discussed in more detail in 6. However, DGCA's of arbitrary degree do play an important role, too. Notice that a DGCA of non-positive degree is in particular a cochain complex of non-positive degree. But that is the same as a chain complex of non-negative degree.

The following is a very simple but important example of a DGCA in non-positive degree.

**Definition 6 (superpoint)** *The “algebra of functions on the superpoint” is the DGCA*

$$C(\mathbf{pt}) := (\mathbb{R} \oplus \mathbb{R}[-1], d_{\mathbf{pt}}) \quad (46)$$

where the product on  $\mathbb{R} \oplus \mathbb{R}[-1]$  is the tensor product over  $\mathbb{R}$ , and where the differential  $d_{\mathbf{pt}} : \mathbb{R}[-1] \rightarrow \mathbb{R}$  is the canonical isomorphism.

The smooth space associated to this algebra according to definition 3 is just the ordinary point, because for any test domain  $U$  the set

$$\mathrm{Hom}_{\mathrm{DGCA's}}(C(\mathbf{pt}), \Omega^\bullet(U)) \quad (47)$$

contains only the morphism which sends  $1 \in \mathbb{R}$  to the constant unit function on  $U$ , and which sends  $\mathbb{R}[-1]$  to 0. However, as is well known from the theory of supermanifolds, the algebra  $C(\mathbf{pt})$  is important in that morphisms from any other DGCA  $A$  into it compute the (shifted) *tangent space* corresponding to  $A$ . From our point of view here this manifests itself in particular by the fact that for  $X$  any manifold, we have a canonical injection

$$\Omega^\bullet(TX) \hookrightarrow \Omega^\bullet(\mathrm{maps}(C^\infty(X), C(\mathbf{pt}))) \quad (48)$$

of the differential forms on the tangent bundle of  $X$  into the differential forms on the smooth space of algebra homomorphisms of  $C^\infty(X)$  to  $C(\mathbf{pt})$ :

for every test domain  $U$  an element in  $\mathrm{Hom}_{\mathrm{DGCA's}}(C^\infty(X), C(\mathbf{pt} \otimes \Omega^\bullet(U)))$  comes from a pair consisting of a smooth map  $f : U \rightarrow X$  and a vector field  $v \in \Gamma(TX)$ . Together this constitutes a smooth map  $\hat{f} : U \rightarrow TX$  and hence for every form  $\omega \in \Omega^\bullet(TX)$  we obtain a form on  $\mathrm{maps}(C^\infty(X), C(\mathbf{pt}))$  by the assignment

$$((f, v) \in \mathrm{Hom}_{\mathrm{DGCA's}}(C^\infty(X), C(\mathbf{pt} \otimes \Omega^\bullet(U)))) \mapsto (\hat{f}^* \omega \in \Omega^\bullet(U)) \quad (49)$$

over each test domain  $U$ .

In 6.1.1 we discuss how in the analogous fashion we obtain the Weil algebra  $W(\mathfrak{g})$  of any  $L_\infty$ -algebra  $\mathfrak{g}$  from its Chevalley-Eilenberg algebra  $CE(\mathfrak{g})$  by mapping that to  $C(\mathbf{pt})$ . This says that the Weil algebra is like the space of functions on the shifted tangent bundle of the “space” that the Chevalley-Eilenberg algebra is the space of functions on. See also figure 3.

## 5.2 Homotopies and inner derivations

When we forget the algebra structure of DGCA's, they are simply cochain complexes. As such they naturally live in a 2-category  $\mathrm{Ch}^\bullet$  whose objects are cochain complexes  $(V^\bullet, d_V)$ , whose morphisms

$$(V^\bullet, d_V) \xleftarrow{f^*} (W^\bullet, d_W) \quad (50)$$

are degree preserving linear maps  $V^\bullet \xleftarrow{f^*} W^\bullet$  that do respect the differentials,

$$[d, f^*] := d_V \circ f^* - f^* \circ d_W = 0, \quad (51)$$

and whose 2-morphisms

$$\begin{array}{ccc} & f^* & \\ & \curvearrowright & \\ (V^\bullet, d_V) & \begin{array}{c} \Downarrow \rho \\ \Downarrow \end{array} & (W^\bullet, d_W) \\ & \curvearrowleft & \\ & g^* & \end{array} \quad (52)$$

are cochain homotopies, namely linear degree -1 maps  $\rho : W^\bullet \rightarrow V^\bullet$  with the property that

$$g^* = f^* + [d, \rho] = f^* + d_V \circ \rho + \rho \circ d_W . \quad (53)$$

Later in 6.2 we will also look at morphisms that do preserve the algebra structure, and homotopies of these. Notice that we can compose a 2-morphism from left and right with 1-morphisms, to obtain another 2-morphism

$$\begin{array}{ccccc} & & f^* & & \\ & \swarrow & \parallel & \searrow & \\ (U^\bullet, d_U) & \xleftarrow{h^*} & (V^\bullet, d_V) & \xrightarrow{\rho} & (W^\bullet, d_W) \xleftarrow{j^*} (X^\bullet, d_X) \\ & \nwarrow & \parallel & \swarrow & \\ & & g^* & & \end{array} \quad (54)$$

whose component map now is

$$h^* \circ \rho \circ j^* : X^\bullet \xrightarrow{j^*} W^\bullet \xrightarrow{\rho} V^\bullet \xrightarrow{h^*} U^\bullet . \quad (55)$$

This will be important for the interpretation of the diagrams we discuss, of the type 64 and 70 below.

Of special importance are linear endomorphisms  $V^\bullet \xleftarrow{\rho} V^\bullet$  of DGCA's which are algebra derivations. Among them, the *inner* derivations in turn play a special role:

**Definition 7 (inner derivations)** *On any DGCA  $(V^\bullet, d_V)$ , a degree 0 endomorphism*

$$(V^\bullet, d_V) \xleftarrow{L} (V^\bullet, d_V) \quad (56)$$

*is called an inner derivation if*

- *it is an algebra derivation of degree 0;*
- *it is connected to the 0-derivation, i.e. there is a 2-morphisms*

$$\begin{array}{ccc} & 0 & \\ & \swarrow & \searrow \\ (V^\bullet, d_V) & \xleftarrow{\rho} & (V^\bullet, d_V) \\ & \nwarrow & \swarrow \\ & L=[d_V, \rho] & \end{array} , \quad (57)$$

*where  $\rho$  comes from an algebra derivation of degree -1.*

**Remark.** Inner derivations generalize the notion of a Lie derivative on differential forms, and hence they encode the notion of vector fields in the context of DGCA's.

### 5.2.1 Examples

**Lie derivatives on ordinary differential forms.** The formula sometimes known as “*Cartan’s magic formula*”, which says that on a smooth space  $Y$  the Lie derivative  $L_v \omega$  of a differential form  $\omega \in \Omega^\bullet(Y)$  along a vector field  $v \in \Gamma(TY)$  is given by

$$L_v \omega = [d, \iota_v] , \quad (58)$$

where  $\iota_v : \Omega^\bullet(Y) \rightarrow \Omega^\bullet(Y)$ , says that Lie derivatives on differential forms are inner derivations, in our sense. When  $Y$  is equipped with a smooth projection  $\pi : Y \rightarrow X$ , it is of importance to distinguish the vector fields vertical with respect to  $\pi$ . The abstract formulation of this, applicable to arbitrary DGCA's, is given in 5.3 below.

### 5.3 Vertical flows and basic forms

We will prominently be dealing with surjections

$$A \xleftarrow{i^*} B \quad (59)$$

of differential graded commutative algebras that play the role of the dual of an injection

$$F \xleftarrow{i} P \quad (60)$$

of a fiber into a bundle. We need a way to identify in the context of DGCAs which inner derivations of  $P$  are *vertical* with respect to  $i$ . Then we can find the algebra corresponding to the *basis* of  $P$  as those elements of  $B$  which are annihilated by all vertical derivations.

**Definition 8 (vertical derivations)** *Given any surjection of differential graded algebras*

$$F \xleftarrow{i^*} P \quad (61)$$

*we say that the vertical inner derivations*

$$\begin{array}{ccc} & 0 & \\ \curvearrowright & \Downarrow \rho & \curvearrowleft \\ P & & P \\ \curvearrowleft & [d_P, \rho] & \curvearrowright \end{array} \quad (62)$$

(this diagram is in the category of cochain complexes, compare the beginning of 6.2) on  $P$  with respect to  $i^*$  are those inner derivations

- for which there exists an inner derivation of  $F$

$$\begin{array}{ccc} & 0 & \\ \curvearrowright & \Downarrow \rho' & \curvearrowleft \\ F & & F \\ \curvearrowleft & [d_P, \rho'] & \curvearrowright \end{array} \quad (63)$$

such that

$$\begin{array}{ccc} & 0 & \\ \curvearrowright & \Downarrow \rho' & \curvearrowleft \\ F & & F \\ \curvearrowleft & [d, \rho'] & \curvearrowright \\ \uparrow & & \uparrow \\ & 0 & \\ \curvearrowright & \Downarrow \rho & \curvearrowleft \\ P & & P \\ \curvearrowleft & [d, \rho] & \curvearrowright \end{array} ; \quad (64)$$

- and where  $\rho'$  is a contraction,  $\rho' = \iota_x$ , i.e. a derivation which sends indecomposables to degree 0.

**Definition 9 (basic elements)** *Given any surjection of differential graded algebras*

$$F \xleftarrow{i^*} P \quad (65)$$

*we say that the algebra*

$$P_{\text{basic}} = \bigcap_{\rho \text{ vertical}} \ker(\rho) \cap \ker(\rho \circ d_P) \quad (66)$$

*of basic elements of  $P$  (with respect to the surjection  $i^*$ ) is the subalgebra of  $P$  of all those elements  $a \in P$  which are annihilated by all  $i^*$ -vertical derivations  $\rho$ , in that*

$$\rho(a) = 0 \quad (67)$$

$$\rho(d_P a) = 0. \quad (68)$$

We have a canonical inclusion

$$P \xleftarrow{p^*} P_{\text{basic}}. \quad (69)$$

Diagrammatically the above condition says that

$$\begin{array}{ccc} & 0 & \\ & \Downarrow \rho' & \\ F & \xleftarrow{i^*} & F \\ & \Downarrow [d, \rho'] & \\ P & \xleftarrow{i^*} & P \\ & \Downarrow \rho & \\ & \Downarrow [d, \rho] & \\ P_{\text{basic}} & \xleftarrow{p^*} & P_{\text{basic}} \end{array} \quad (70)$$

### 5.3.1 Examples

**Basic forms on a bundle** As a special case of the above general definition, we reobtain the standard notion of basic differential forms on a smooth surjective submersion  $\pi : Y \rightarrow X$  with connected fibers.

**Definition 10** *Let  $\pi : Y \rightarrow X$  be a smooth map. The **vertical deRham complex**,  $\Omega_{\text{vert}}^\bullet(Y)$ , with respect to  $Y$  is the deRham complex of  $Y$  modulo those forms that vanish when restricted in all arguments to vector fields in the kernel of  $\pi_* : \Gamma(TY) \rightarrow \Gamma(TX)$ , namely to vertical vector fields.*

The induced differential on  $\Omega_{\text{vert}}^\bullet(Y)$  sends  $\omega_{\text{vert}} = i^* \omega$  to

$$d_{\text{vert}} : i^* \omega \mapsto i^* d\omega. \quad (71)$$

**Proposition 2** *This is well defined. The quotient  $\Omega_{\text{vert}}^\bullet(Y)$  with the differential induced from  $\Omega^\bullet(Y)$  is indeed a dg-algebra, and the projection*

$$\Omega_{\text{vert}}^\bullet(Y) \xleftarrow{i^*} \Omega^\bullet(Y) \quad (72)$$

*is a homomorphism of dg-algebras (in that it does respect the differential).*

Proof. Notice that if  $\omega \in \Omega^\bullet(Y)$  vanishes when evaluated on vertical vector fields then obviously so does  $\alpha \wedge \omega$ , for any  $\alpha \in \Omega^\bullet(Y)$ . Moreover, due to the formula

$$d\omega(v_1, \dots, v_{n+1}) = \sum_{\sigma \in \text{Sh}(1, n+1)} \pm v_{\sigma_1} \omega(v_{\sigma_2}, \dots, v_{\sigma_{n+1}}) + \sum_{\sigma \in \text{Sh}(2, n+1)} \pm \omega([v_{\sigma_1}, v_{\sigma_2}], v_{\sigma_3}, \dots, v_{\sigma_{n+1}}) \quad (73)$$

and the fact that for  $v, w$  vertical so is  $[v, w]$  and hence  $d\omega$  is also vertical. This gives that vertical differential forms on  $Y$  form a dg-subalgebra of the algebra of all forms on  $Y$ . Therefore if  $i^* \omega = i^* \omega'$  then

$$di^* \omega' = i^* d\omega' = i^* d(\omega + (\omega' - \omega)) = i^* d\omega + 0 = di^* \omega. \quad (74)$$

Hence the differential is well defined and  $i^*$  is then, by construction, a morphism of dg-algebras.  $\square$

Recall the following standard definition of basic differential forms.

**Definition 11 (basic forms)** *Given a surjective submersion  $\pi : Y \rightarrow X$ , the basic forms on  $Y$  are those with the property that they and their differentials are annihilated by all vertical vector fields*

$$\omega \in \Omega^\bullet(Y)_{\text{basic}} \Leftrightarrow \forall v \in \ker(\pi) : \iota_v \omega = \iota_v d\omega = 0. \quad (75)$$

It is a standard result that

**Proposition 3** *If  $\pi : Y \rightarrow X$  is locally trivial and has connected fibers, then the basic forms are precisely those coming from pullback along  $\pi$*

$$\Omega^\bullet(Y)_{\text{basic}} \simeq \Omega^\bullet(X). \quad (76)$$

**Remark.** The reader should compare this situation with the definition of invariant polynomials in 6.3.

The next proposition asserts that these statements about ordinary basic differential forms are indeed a special case of the general definition of basic elements with respect to a surjection of DGCAs, definition 9.

**Proposition 4** *Given a surjective submersion  $\pi : Y \rightarrow X$  with connected fibers, then*

- the inner derivations of  $\Omega^\bullet(Y)$  which are vertical with respect to  $\Omega_{\text{vert}}^\bullet(Y) \xleftarrow{i^*} \Omega^\bullet(Y)$  according to the general definition 8, come precisely from contractions  $\iota_v$  with vertical vector fields  $v \in \ker(\pi_*) \subset \Gamma(TY)$ ;
- the basic differential forms on  $Y$  according to definition 11 coincide with the basic elements of  $\Omega^\bullet(Y)$  relative to the above surjection

$$\Omega_{\text{vert}}^\bullet(Y) \xleftarrow{i^*} \Omega^\bullet(Y) \quad (77)$$

according to the general definition 9.

Proof. We first show that if  $\Omega^\bullet(Y) \xleftarrow{\rho} \Omega^\bullet(Y)$  is a vertical algebra derivation, then  $\rho$  has to annihilate all forms in the image of  $\pi_*$ . Let  $\alpha \in \Omega^\bullet(Y)$  be any 1-form and  $\omega = \pi^* \beta$  for  $\beta \in \Omega^1(X)$ . Then the wedge product  $\alpha \wedge \omega$  is annihilated by the projection to  $\Omega_{\text{vert}}^\bullet(Y)$  and we find

$$\begin{array}{ccc} \rho(\omega) \wedge \alpha & \xleftarrow{\quad} & 0 \\ \uparrow i^* & & \uparrow i^* \\ \rho(\omega) \wedge \alpha + \rho(\alpha) \wedge \omega & \xleftarrow{\rho} & \alpha \wedge \omega \end{array} \quad (78)$$



We see that  $\rho(\omega) \wedge \alpha$  has to vanish for all  $\alpha$ . Therefore  $\rho(\omega)$  has to vanish for all  $\omega$  pulled back along  $\pi^*$ . Hence  $\rho$  must be contraction with a vertical vector field. It then follows from the condition 64 that a basic form is one annihilated by all such  $\rho$  and all such  $\rho \circ d$ .  $\square$

Possibly the most familiar kinds of surjective submersions are

- Fiber bundles.

Indeed, the standard Cartan-Ehresmann theory of connections of principal bundles is obtained in our context by fixing a Lie group  $G$  and a principal  $G$ -bundle  $p : P \rightarrow X$  and then using  $Y = P$  itself as the surjective submersion. The definition of a connection on  $P$  in terms of a  $\mathfrak{g}$ -valued 1-form on  $P$  can be understood as the descent data for a connection on  $P$  obtained with respect to canonical trivialization of the pullback of  $P$  to  $Y = P$ . Using for the surjective submersion  $Y$  a principal  $G$ -bundle  $P \rightarrow X$  is also most convenient for studying all kinds of higher  $n$ -bundles obstructing lifts of the given  $G$ -bundle. This is why we will often make use of this choice in the following.

- Covers by open subsets.

The disjoint union of all sets in a cover of  $X$  by open subsets of  $X$  forms a surjective submersion  $\pi : Y \rightarrow X$ . In large parts of the literature on descent (locally trivialized bundles), these are the only kinds of surjective submersions that are considered. We will find here, that in order to characterize principal  $n$ -bundles entirely in terms of  $L_\infty$ -algebraic data, open covers are too restrictive and the full generality of surjective submersions is needed. The reason is that, for  $\pi : Y \rightarrow X$  a cover by open subsets, there are no nontrivial vertical vector fields

$$\ker(\pi) = 0, \quad (79)$$

hence

$$\Omega_{\text{vert}}^\bullet(Y) = 0. \quad (80)$$

With the definition of  $\mathfrak{g}$ -descent objects in 7.1 this implies that all  $\mathfrak{g}$ -descent objects over a cover by open subsets are trivial.

There are two important subclasses of surjective submersions  $\pi : Y \rightarrow X$ :

- those for which  $Y$  is (smoothly) contractible;
- those for which the fibers of  $Y$  are connected.

We say  $Y$  is (smoothly) contractible if the identity map  $\text{Id} : Y \rightarrow Y$  is (smoothly) homotopic to a map  $Y \rightarrow Y$  which is constant on each connected component. Hence  $Y$  is a disjoint union of spaces that are each (smoothly) contractible to a point. In this case the Poincaré lemma says that the dg-algebra  $\Omega^\bullet(Y)$  of differential forms on  $Y$  is contractible; each closed form is exact:

$$\begin{array}{ccc} & 0 & \\ & \Downarrow \tau & \\ \Omega^\bullet(Y) & & \Omega^\bullet(Y) \end{array} \quad (81)$$

$[d, \tau]$

Here  $\tau$  is the familiar homotopy operator that appears in the proof of the Poincaré lemma. In practice, we often make use of the best of both worlds: surjective submersions that are (smoothly) contractible to a discrete set but still have a sufficiently rich collection of vertical vector fields. The way to obtain these is by refinement: starting with any surjective submersion  $\pi : Y \rightarrow X$  which has good vertical vector fields

but might not be contractible, we can cover  $Y$  itself with open balls, whose disjoint union,  $Y'$ , then forms a surjective submersion  $Y' \rightarrow Y$  over  $Y$ . The composite  $\pi'$

$$\begin{array}{ccc} Y' & \xrightarrow{\quad} & Y \\ & \searrow \pi & \swarrow \\ & X & \end{array} \quad (82)$$

is then a contractible surjective submersion of  $X$ . We will see that all our descent objects can be pulled back along refinements of surjective submersions this way, so that it is possible, without restriction of generality, to always work on contractible surjective submersions. Notice that for these the structure of

$$\Omega_{\text{vert}}^{\bullet}(Y) \longleftarrow \Omega^{\bullet}(Y) \longleftarrow \Omega^{\bullet}(X) \quad (83)$$

is rather similar to that of

$$\text{CE}(\mathfrak{g}) \longleftarrow W(\mathfrak{g}) \longleftarrow \text{inv}(\mathfrak{g}) , \quad (84)$$

since  $W(\mathfrak{g})$  is also contractible, according to proposition 6.

**Vertical derivations on universal  $\mathfrak{g}$ -bundles.** The other important example of vertical flows, those on DGAs modelling universal  $\mathfrak{g}$ -bundle for  $\mathfrak{g}$  an  $L_{\infty}$ -algebra, is discussed at the beginning of 6.3.

## 6 $L_{\infty}$ -algebras and their String-like extensions

$L_{\infty}$ -algebras are a generalization of Lie algebras, where the Jacobi identity is demanded to hold only up to higher coherent equivalence, as the category theorist would say, or “strongly homotopic”, as the homotopy theorist would say.

### 6.1 $L_{\infty}$ -algebras

**Definition 12** *Given a graded vector space  $V$ , the tensor space  $T^{\bullet}(V) := \bigoplus_{n=0} V^{\otimes n}$  with  $V^0$  being the ground field. We will denote by  $T^a(V)$  the tensor algebra with the concatenation product on  $T^{\bullet}(V)$ :*

$$x_1 \otimes x_2 \otimes \cdots \otimes x_p \bigotimes x_{p+1} \otimes \cdots \otimes x_n \mapsto x_1 \otimes x_2 \otimes \cdots \otimes x_n \quad (85)$$

and by  $T^c(V)$  the tensor coalgebra with the deconcatenation product on  $T^{\bullet}(V)$ :

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto \sum_{p+q=n} x_1 \otimes x_2 \otimes \cdots \otimes x_p \bigotimes x_{p+1} \otimes \cdots \otimes x_n. \quad (86)$$

The graded symmetric algebra  $\wedge^{\bullet}(V)$  is the quotient of the tensor algebra  $T^a(V)$  by the graded action of the symmetric groups  $\mathbf{S}_n$  on the components  $V^{\otimes n}$ . The graded symmetric coalgebra  $\vee^{\bullet}(V)$  is the sub-coalgebra of the tensor coalgebra  $T^c(V)$  fixed by the graded action of the symmetric groups  $\mathbf{S}_n$  on the components  $V^{\otimes n}$ .

**Remark.**  $\vee^{\bullet}(V)$  is spanned by graded symmetric tensors

$$x_1 \vee x_2 \vee \cdots \vee x_p \quad (87)$$

for  $x_i \in V$  and  $p \geq 0$ , where we use  $\vee$  rather than  $\wedge$  to emphasize the coalgebra aspect, e.g.

$$x \vee y = x \otimes y \pm y \otimes x. \quad (88)$$

In characteristic zero, the graded symmetric algebra can be identified with a sub-algebra of  $T^a(V)$  but that is unnatural and we will try to avoid doing so. The coproduct on  $\vee^\bullet(V)$  is given by

$$\Delta(x_1 \vee x_2 \cdots \vee x_n) = \sum_{p+q=n} \sum_{\sigma \in \text{Sh}(p,q)} \epsilon(\sigma)(x_{\sigma(1)} \vee x_{\sigma(2)} \cdots x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \vee \cdots x_{\sigma(n)}). \quad (89)$$

The notation here means the following:

- $\text{Sh}(p, q)$  is the subset of all those bijections (the “unshuffles”) of  $\{1, 2, \dots, p+q\}$  that have the property that  $\sigma(i) < \sigma(i+1)$  whenever  $i \neq p$ ;
- $\epsilon(\sigma)$ , which is shorthand for  $\epsilon(\sigma, x_1 \vee x_2, \dots, x_{p+q})$ , the Koszul sign, defined by

$$x_1 \vee \cdots \vee x_n = \epsilon(\sigma) x_{\sigma(1)} \vee \cdots x_{\sigma(n)}. \quad (90)$$

**Definition 13 ( $L_\infty$ -algebra)** An  $L_\infty$ -algebra  $\mathfrak{g} = (\mathfrak{g}, D)$  is a  $\mathbb{N}_+$ -graded vector space  $\mathfrak{g}$  equipped with a degree -1 coderivation

$$D : \vee^\bullet \mathfrak{g} \rightarrow \vee^\bullet \mathfrak{g} \quad (91)$$

on the graded co-commutative coalgebra generated by  $\mathfrak{g}$ , such that  $D^2 = 0$ . This induces a differential

$$d_{\text{CE}(\mathfrak{g})} : \text{Sym}^\bullet(\mathfrak{g}) \rightarrow \text{Sym}^{\bullet+1}(\mathfrak{g}) \quad (92)$$

on graded-symmetric multilinear functions on  $\mathfrak{g}$ . When  $\mathfrak{g}$  is finite dimensional this yields a degree +1 differential

$$d_{\text{CE}(\mathfrak{g})} : \wedge^\bullet \mathfrak{g}^* \rightarrow \wedge^{\bullet+1} \mathfrak{g}^* \quad (93)$$

on the graded-commutative algebra generated from  $\mathfrak{g}^*$ . This is the Chevalley-Eilenberg dg-algebra corresponding to the  $L_\infty$ -algebra  $\mathfrak{g}$ .

**Remark.** That the original definition of  $L_\infty$ -algebras in terms of multibrackets yields a codifferential coalgebra as above was shown in [57]. That every such codifferential comes from a collection of multibrackets this way is due to [58].

**Example** For  $(\mathfrak{g}[-1], [\cdot, \cdot])$  an ordinary Lie algebra (meaning that we regard the vector space  $\mathfrak{g}$  to be in degree 1), the corresponding Chevalley-Eilenberg qDGCA is

$$\text{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*, d_{\text{CE}(\mathfrak{g})}) \quad (94)$$

with

$$d_{\text{CE}(\mathfrak{g})} : \mathfrak{g}^* \xrightarrow{[\cdot, \cdot]^*} \mathfrak{g}^* \wedge \mathfrak{g}^*. \quad (95)$$

If we let  $\{t_a\}$  be a basis of  $\mathfrak{g}$  and  $\{C^a_{bc}\}$  the corresponding structure constants of the Lie bracket  $[\cdot, \cdot]$ , and if we denote by  $\{t^a\}$  the corresponding basis of  $\mathfrak{g}^*$ , then we get

$$d_{\text{CE}(\mathfrak{g})} t^a = -\frac{1}{2} C^a_{bc} t^b \wedge t^c. \quad (96)$$

If  $\mathfrak{g}$  is concentrated in degree 1,  $\dots, n$ , we also say that  $\mathfrak{g}$  is a **Lie  $n$ -algebra**. Notice that built in we have a shift of degree for convenience, which makes ordinary Lie 1-algebras be in degree 1 already. In much of the literature a Lie  $n$ -algebra would be based on a vector space concentrated in degrees 0 to  $n-1$ . An ordinary Lie algebra is a Lie 1-algebra. Here the coderivation differential  $D = [\cdot, \cdot]$  is just the Lie bracket, extended as a coderivation to  $\vee^\bullet \mathfrak{g}$ , with  $\mathfrak{g}$  regarded as being in degree 1.

In the rest of the paper we assume, just for simplicity and since it is sufficient for our applications, all  $\mathfrak{g}$  to be finite-dimensional. Then, by the above, these  $L_\infty$ -algebras are equivalently conceived of in terms of

their dual Chevalley-Eilenberg algebras,  $\mathrm{CE}(\mathfrak{g})$ , as indeed every quasi-free differential graded commutative algebra (“qDGCA”, meaning that it is free as a graded commutative algebra) corresponds to an  $L_\infty$ -algebra. We will find it convenient to work entirely in terms of qDGCA’s, which we will usually denote as  $\mathrm{CE}(\mathfrak{g})$ .

While not very interesting in themselves, truly free differential algebras are a useful tool for handling quasi-free differential algebras.

**Definition 14** *We say a qDGCA is free (even as a differential algebra) if it is of the form*

$$F(V) := (\wedge^\bullet(V^* \oplus V^*[1]), d_{F(V)}) \quad (97)$$

with

$$d_{F(V)}|_{V^*} = \sigma : V^* \rightarrow V^*[1] \quad (98)$$

the canonical isomorphism and

$$d_{F(V)}|_{V^*[1]} = 0. \quad (99)$$

**Remark.** Such algebras are indeed free in that they satisfy the universal property: given any linear map  $V \rightarrow W$ , it uniquely extends to a morphism of qDGCA’s  $F(V) \rightarrow (\wedge^\bullet(W^*), d)$  for any choice of differential  $d$ .

**Example.** The free qDGCA on a 1-dimensional vector space in degree 0 is the graded commutative algebra freely generated by two generators,  $t$  of degree 0 and  $dt$  of degree 1, with the differential acting as  $d : t \mapsto dt$  and  $d : dt \mapsto 0$ . In rational homotopy theory, this models the interval  $I = [0, 1]$ . The fact that the qDGCA is free corresponds to the fact that the interval is homotopy equivalent to the point.

We will be interested in qDGCA’s that arise as mapping cones of morphisms of  $L_\infty$ -algebras.

**Definition 15 (“mapping cone” of qDGCA’s)** *Let*

$$\mathrm{CE}(\mathfrak{h}) \xleftarrow{t^*} \mathrm{CE}(\mathfrak{g}) \quad (100)$$

*be a morphism of qDGCA’s. The mapping cone of  $t^*$ , which we write  $\mathrm{CE}(\mathfrak{h} \xrightarrow{t^*} \mathfrak{g})$ , is the qDGCA whose underlying graded algebra is*

$$\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1]) \quad (101)$$

*and whose differential  $d_{t^*}$  is such that it acts as*

$$d_{t^*} = \begin{pmatrix} d_{\mathfrak{g}} & 0 \\ t^* & d_{\mathfrak{h}} \end{pmatrix}. \quad (102)$$

We postpone a more detailed definition and discussion to 8.1; see definition 39 and proposition 36. Strictly speaking, the more usual notion of mapping cones of chain complexes applies to  $t : \mathfrak{h} \rightarrow \mathfrak{g}$ , but then is extended as a derivation differential to the entire qDGCA.

**Definition 16 (Weil algebra of an  $L_\infty$ -algebra)** *The mapping cone of the identity on  $\mathrm{CE}(\mathfrak{g})$  is the Weil algebra*

$$\mathrm{W}(\mathfrak{g}) := \mathrm{CE}(\mathfrak{g} \xrightarrow{\mathrm{Id}} \mathfrak{g}) \quad (103)$$

*of  $\mathfrak{g}$ .*

**Proposition 5** *For  $\mathfrak{g}$  an ordinary Lie algebra this does coincide with the ordinary Weil algebra of  $\mathfrak{g}$ .*

*Proof.* See the example in 6.1.1. □

The Weil algebra has two important properties.

**Proposition 6** *The Weil algebra  $W(\mathfrak{g})$  of any  $L_\infty$ -algebra  $\mathfrak{g}$*

- *is isomorphic to a free differential algebra*

$$W(\mathfrak{g}) \simeq F(\mathfrak{g}), \quad (104)$$

*and hence is contractible;*

- *has a canonical surjection*

$$CE(\mathfrak{g}) \xleftarrow{i^*} W(\mathfrak{g}). \quad (105)$$

Proof. Define a morphism

$$f : F(\mathfrak{g}) \rightarrow W(\mathfrak{g}) \quad (106)$$

by setting

$$f : a \mapsto a \quad (107)$$

$$f : (d_{F(V)}a = \sigma a) \mapsto (d_{W(\mathfrak{g})}a = d_{CE(\mathfrak{g})}a + \sigma a) \quad (108)$$

for all  $a \in \mathfrak{g}^*$  and extend as an algebra homomorphism. This clearly respects the differentials: for all  $a \in V^*$  we find

$$\begin{array}{ccc} a & \xrightarrow{d_{F(\mathfrak{g})}} & \sigma a \\ \downarrow f & & \downarrow f \\ a & \xrightarrow{d_{W(\mathfrak{g})}} & d_{CE(\mathfrak{g})}a + \sigma a \end{array} \quad \text{and} \quad \begin{array}{ccc} \sigma a & \xrightarrow{d_{F(\mathfrak{g})}} & 0 \\ \downarrow f & & \downarrow f \\ d_{W(\mathfrak{g})}a & \xrightarrow{d_{W(\mathfrak{g})}} & 0 \end{array}. \quad (109)$$

One checks that the strict inverse exists and is given by

$$f^{-1}|_{\mathfrak{g}^*} : a \mapsto a \quad (110)$$

$$f^{-1}|_{\mathfrak{g}^*[1]} : \sigma a \mapsto d_{F(\mathfrak{g})}a - d_{CE(\mathfrak{g})}a. \quad (111)$$

Here  $\sigma : \mathfrak{g}^* \rightarrow \mathfrak{g}^*[1]$  is the canonical isomorphism that shifts the degree. The surjection  $CE(\mathfrak{g}) \xleftarrow{i^*} W(\mathfrak{g})$  simply projects out all elements in the shifted copy of  $\mathfrak{g}$ :

$$i^*|_{\wedge^\bullet \mathfrak{g}^*} = \text{id} \quad (112)$$

$$i^*|_{\mathfrak{g}^*[1]} = 0. \quad (113)$$

This is an algebra homomorphism that respects the differential. □

As a corollary we obtain

**Corollary 1** *For  $\mathfrak{g}$  any  $L_\infty$ -algebra, the cohomology of  $W(\mathfrak{g})$  is trivial.*

**Proposition 7** *The step from a Chevalley-Eilenberg algebra to the corresponding Weil algebra is functorial: for any morphism*

$$CE(\mathfrak{h}) \xleftarrow{f^*} CE(\mathfrak{g}) \quad (114)$$

*we obtain a morphism*

$$W(\mathfrak{h}) \xleftarrow{\hat{f}^*} W(\mathfrak{g}) \quad (115)$$

*and this respects composition.*

Proof. The morphism  $\hat{f}^*$  acts as for all generators  $a \in \mathfrak{g}^*$  as

$$\hat{f}^* : a \mapsto f^*(a) \quad (116)$$

and

$$\hat{f}^* : \sigma a \mapsto \sigma f^*(a). \quad (117)$$

We check that this does repeat the differentials

$$\begin{array}{ccc} a & \xrightarrow{d_{W(\mathfrak{g})}} & d_{CE(\mathfrak{g})}a + \sigma a \\ \downarrow \hat{f}^* & & \downarrow \hat{f}^* \\ f^*(a) & \xrightarrow{d_{W(\mathfrak{h})}} & d_{CE(\mathfrak{h})}f^*(a) + \sigma f^*(a) \end{array} \quad , \quad \begin{array}{ccc} \sigma a & \xrightarrow{d_{W(\mathfrak{g})}} & -\sigma(d_{CE(\mathfrak{g})}a) \\ \downarrow \hat{f}^* & & \downarrow \hat{f}^* \\ \sigma f^*(a) & \xrightarrow{d_{W(\mathfrak{h})}} & -\sigma(d_{CE(\mathfrak{h})}a) \end{array} \quad (118)$$

□

**Remark.** As we will shortly see,  $W(\mathfrak{g})$  plays the role of the algebra of differential forms on the universal  $\mathfrak{g}$ -bundle. The surjection  $CE(\mathfrak{g}) \xleftarrow{i^*} W(\mathfrak{g})$  plays the role of the restriction to the differential forms on the fiber of the universal  $\mathfrak{g}$ -bundle.

### 6.1.1 Examples

In section 6.4 we construct large families of examples of  $L_\infty$ -algebras, based on the first two of the following examples:

**1. Ordinary Weil algebras as Lie 2-algebras.** What is ordinarily called the Weil algebra  $W(\mathfrak{g})$  of a Lie algebra  $(\mathfrak{g}[-1], [\cdot, \cdot])$  can, since it is again a DGCA, also be interpreted as the Chevalley-Eilenberg algebra of a Lie 2-algebra. This Lie 2-algebra we call  $\text{inn}(\mathfrak{g})$ . It corresponds to the Lie 2-group  $\text{INN}(G)$  discussed in [65]:

$$W(\mathfrak{g}) = CE(\text{inn}(\mathfrak{g})). \quad (119)$$

We have

$$W(\mathfrak{g}) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1]), d_{W(\mathfrak{g})}). \quad (120)$$

Denoting by  $\sigma : \mathfrak{g}^* \rightarrow \mathfrak{g}^*[1]$  the canonical isomorphism, extended as a derivation to all of  $W(\mathfrak{g})$ , we have

$$d_{W(\mathfrak{g})} : \mathfrak{g}^* \xrightarrow{[\cdot, \cdot]^* + \sigma} \mathfrak{g}^* \wedge \mathfrak{g}^* \oplus \mathfrak{g}^*[1] \quad (121)$$

and

$$d_{W(\mathfrak{g})} : \mathfrak{g}^*[1] \xrightarrow{-\sigma \circ d_{CE(\mathfrak{g})} \circ \sigma^{-1}} \mathfrak{g}^* \otimes \mathfrak{g}^*[1]. \quad (122)$$

With  $\{t^a\}$  a basis for  $\mathfrak{g}^*$  as above, and  $\{\sigma t^a\}$  the corresponding basis of  $\mathfrak{g}^*[1]$  we find

$$d_{W(\mathfrak{g})} : t^a \mapsto -\frac{1}{2}C^a_{bc}t^b \wedge t^c + \sigma t^a \quad (123)$$

and

$$d_{W(\mathfrak{g})} : \sigma t^a \mapsto -C^a_{bc}t^b \sigma t^c. \quad (124)$$

The Lie 2-algebra  $\text{inn}(\mathfrak{g})$  is, in turn, nothing but the strict Lie 2-algebra as in the third example below, which comes from the infinitesimal crossed module  $(\mathfrak{g} \xrightarrow{\text{Id}} \mathfrak{g} \xrightarrow{\text{ad}} \text{der}(\mathfrak{g}))$ .

**2. Shifted  $\mathfrak{u}(1)$ .** By the above, the qDGCA corresponding to the Lie algebra  $\mathfrak{u}(1)$  is simply

$$\mathrm{CE}(\mathfrak{u}(1)) = (\wedge^\bullet \mathbb{R}[1], d_{\mathrm{CE}(\mathfrak{u}(1))} = 0). \quad (125)$$

We write

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) = (\wedge^\bullet \mathbb{R}[n], d_{\mathrm{CE}(b^{n-1}\mathfrak{u}(1))} = 0) \quad (126)$$

for the Chevalley-Eilenberg algebras corresponding to the Lie  $n$ -algebras  $b^{n-1}\mathfrak{u}(1)$ .

**3. Infinitesimal crossed modules and strict Lie 2-algebras.** An *infinitesimal crossed module* is a diagram

$$(\mathfrak{h} \xrightarrow{t} \mathfrak{g} \xrightarrow{\alpha} \mathrm{der}(\mathfrak{h})) \quad (127)$$

of Lie algebras where  $t$  and  $\alpha$  satisfy two compatibility conditions. These conditions are equivalent to the nilpotency of the differential on

$$\mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) := (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1]), d_t) \quad (128)$$

defined by

$$d_t|_{\mathfrak{g}^*} = [\cdot, \cdot]_{\mathfrak{g}}^* + t^* \quad (129)$$

$$d_t|_{\mathfrak{h}^*[1]} = \alpha^*, \quad (130)$$

where we consider the vector spaces underlying both  $\mathfrak{g}$  and  $\mathfrak{h}$  to be in degree 1. Here in the last line we regard  $\alpha$  as a linear map  $\alpha : \mathfrak{g} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$ . The Lie 2-algebras  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  thus defined are called strict Lie 2-algebras: these are precisely those Lie 2-algebras whose Chevalley-Eilenberg differential contains at most co-binary components.

**4. Inner derivation  $L_\infty$ -algebras.** In straightforward generalization of the first example we find: for  $\mathfrak{g}$  any  $L_\infty$ -algebra, its Weil algebra  $W(\mathfrak{g})$  is again a DGCA, hence the Chevalley-Eilenberg algebra of some other  $L_\infty$ -algebra. This we address as the  $L_\infty$ -algebra of inner derivations and write

$$\mathrm{CE}(\mathrm{inn}(\mathfrak{g})) := W(\mathfrak{g}). \quad (131)$$

This identification is actually useful for identifying the Lie  $\infty$ -groups that correspond to an integrated picture underlying our differential discussion. In [65] the Lie 3-group corresponding to  $\mathrm{inn}(\mathfrak{g})$  for  $\mathfrak{g}$  the strict Lie 2-algebra of any strict Lie 2-group is discussed. This 3-group is in particular the right codomain for incorporating the the non-fake flat nonabelian gerbes with connection considered in [17] into the integrated version of the picture discussed here. This is indicated in [70] and should be discussed elsewhere.

tangent category	inner automorphism ( $n+1$ )-group	inner derivation Lie ( $n+1$ )-algebra	Weil algebra	shifted tangent bundle
------------------	---------------------------------------	---	--------------	---------------------------

$$\mathrm{CE}(\mathrm{Lie}(TBG)) = \mathrm{CE}(\mathrm{Lie}(\mathrm{INN}(G))) = \mathrm{CE}(\mathrm{inn}(\mathfrak{g})) = W(\mathfrak{g}) = C^\infty(T[1]\mathfrak{g})$$

Figure 3: **A remarkable coincidence of concepts** relates the notion of tangency to the notion of universal bundles. The leftmost equality is discussed in [65]. The second one from the right is the identification 131. The rightmost equality is equation 143.

**Proposition 8** *For  $\mathfrak{g}$  any finite dimensional  $L_\infty$ -algebra, the differential forms on the smooth space of morphisms from the Chevalley-Eilenberg algebra  $\mathrm{CE}(\mathfrak{g})$  to the algebra of “functions on the superpoint”, definition 6, i.e. the elements in  $\mathrm{maps}(\mathrm{CE}(\mathfrak{g}), C(\mathbf{pt}))$ , which come from currents as in definition 5, form the Weil algebra  $W(\mathfrak{g})$  of  $\mathfrak{g}$ :*

$$W(\mathfrak{g}) \subset \mathrm{maps}(\mathrm{CE}(\mathfrak{g}), C(\mathbf{pt})). \quad (132)$$

Proof. For any test domain  $U$ , an element in  $\text{Hom}_{\text{DGCA}s}(\text{CE}(\mathfrak{g}), C(\mathbf{pt}) \otimes \Omega^\bullet(U))$  is specified by a degree 0 algebra homomorphism

$$\lambda : \text{CE}(\mathfrak{g}) \rightarrow \Omega^\bullet(U) \quad (133)$$

and a degree +1 algebra morphism

$$\lambda : \text{CE}(\mathfrak{g}) \rightarrow \Omega^\bullet(U) \quad (134)$$

by

$$a \mapsto \lambda(a) + c \wedge \omega(a) \quad (135)$$

for all  $a \in \mathfrak{g}^*$  and for  $c$  denoting the canonical degree -1 generator of  $C(\mathbf{pt})$ ; such that the equality in the bottom right corner of the diagram

$$\begin{array}{ccc} a & \xrightarrow{d_{\text{CE}(\mathfrak{g})}} & d_{\text{CE}(\mathfrak{g})}a \\ \downarrow & & \downarrow \\ \lambda(a) + c \wedge \omega(a) & \xrightarrow{d_{\mathbf{pt}} + d_U} & \lambda(d_{\text{CE}(\mathfrak{g})}a) + c \wedge \omega(d_{\text{CE}(\mathfrak{g})}a) \\ & & = d(\lambda(a)) + \omega(a) - c \wedge d(\omega(a)) \end{array} \quad (136)$$

holds. Under the two canonical currents on  $C(\mathbf{pt})$  of degree 0 and degree 1, respectively, this gives rise for each  $a \in \mathfrak{g}^*$  of degree  $|a|$  to an  $|a|$ -form and an  $(|a| + 1)$  form on  $\text{maps}(\text{CE}(\mathfrak{g}), C(\mathbf{pt}))$  whose values on a given plot are  $\lambda(a)$  and  $\omega(a)$ , respectively.

By the above diagram, the differential of these forms satisfies

$$d\lambda(a) = \lambda(d_{\text{CE}(\mathfrak{g})}a) + \omega(a) \quad (137)$$

and

$$d\omega(a) = -\omega(d_{\text{CE}(\mathfrak{g})}a). \quad (138)$$

But this is precisely the structure of  $W(\mathfrak{g})$ .  $\square$

To see the last step, it may be helpful to consider this for a simple case in terms of a basis:

let  $\mathfrak{g}$  be an ordinary Lie algebra,  $\{t^a\}$  a basis of  $\mathfrak{g}^*$  and  $\{C^a_{bc}\}$  the corresponding structure constants. Then, using the fact that, since we are dealing with algebra homomorphisms, we have

$$\lambda(t^a \wedge t^b) = \lambda(t^a) \wedge \lambda(t^b) \quad (139)$$

and

$$\omega(t^a \wedge t^b) = c \wedge (\omega(t^a) \wedge \lambda(t^b) - \lambda(t^a) \wedge \omega(t^b)) \quad (140)$$

we find

$$d\lambda(t^a) = -\frac{1}{2}C^a_{bc}\lambda(t^b) \wedge \lambda(t^c) + \omega(t^a) \quad (141)$$

and

$$d\omega(t^a) = -C^a_{bc}\lambda(t^b) \wedge \omega(t^c). \quad (142)$$

This is clearly just the structure of  $W(\mathfrak{g})$ .

**Remark.** As usual, we may think of the superpoint as an “infinitesimal interval”. The above says that the algebra of inner derivations of an  $L_\infty$ -algebra consists of the maps from the infinitesimal interval to the supermanifold on which  $\text{CE}(\mathfrak{g})$  is the “algebra of functions”. On the one hand this tells us that

$$W(\mathfrak{g}) = C^\infty(T[1]\mathfrak{g}) \quad (143)$$



in supermanifold language. On the other hand, this construction is clearly analogous to the corresponding discussion for Lie  $n$ -groups given in [65]: there the 3-group  $\text{INN}(G)$  of inner automorphisms of a strict 2-group  $G$  was obtained by mapping the “fat point” groupoid  $\mathbf{pt} = \{ \bullet \longrightarrow \circ \}$  into  $G$ . As indicated there, this is a special case of a construction of “tangent categories” which mimics the relation between  $\text{inn}(\mathfrak{g})$  and the shifted tangent bundle  $T[1]\mathfrak{g}$  in the integrated world of Lie  $\infty$ -groups. This relation between these concepts is summarized in figure 3.

## 6.2 $L_\infty$ -algebra homotopy and concordance

Like cochain complexes, differential graded algebras can be thought of as being objects in a higher categorical structure, which manifests itself in the fact that there are not only morphisms between DGCAs, but also higher morphisms between these morphisms. It turns out that we need to consider a couple of slightly differing notions of morphisms and higher morphisms for these. While differing, these concepts are closely related among each other, as we shall discuss.

In 5.2 we had already considered 2-morphisms of DGCAs obtained after forgetting their algebra structure and just remembering their differential structure. The 2-morphisms we present now crucially do know about the algebra structure.

	name	nature
infinitesimal	transformation	chain homotopy
		$  \begin{array}{ccc}  & f^* & \\  \text{CE}(\mathfrak{g}) & \begin{array}{c} \Downarrow \eta \\ \Downarrow \end{array} & \text{CE}(\mathfrak{h}) \\  & g^* &   \end{array}  $
finite	homotopy/concordance	extension over interval
		$  \begin{array}{ccccc}  & & g^* & & \\  & \swarrow & & \searrow & \\  \text{CE}(\mathfrak{g}) & \begin{array}{c} \xleftarrow{\text{Id} \otimes s^*} \\ \xrightarrow{\text{Id} \otimes t^*} \end{array} & \text{CE}(\mathfrak{g}) \otimes \Omega^\bullet(I) & \xleftarrow{\eta^*} & \text{CE}(\mathfrak{h}) \\  & \nwarrow & & \nearrow & \\  & & f^* & &   \end{array}  $

Table 5: The two different notions of **higher morphisms** of qDGCAs.

**Infinitesimal homotopies between dg-algebra homomorphisms.** When we restrict attention to cochain maps between qDGCA which respect not only the differentials but also the free graded commutative algebra structure, i.e. to qDGCA homomorphisms, it becomes of interest to express the cochain homotopies in terms of their action on generators of the algebra. We now define transformations (2-morphisms) between morphisms of qDGCA by first defining them for the case when the domain is a Weil algebra, and then extending the definition to arbitrary qDGCA.

**Definition 17 (transformation of morphisms of  $L_\infty$ -algebras)** *We define transformations between qDGCA morphisms in two steps*

- A 2-morphism

$$\begin{array}{ccc}
 & f^* & \\
 \text{CE}(\mathfrak{g}) & \begin{array}{c} \Downarrow \eta \\ \Downarrow \end{array} & \text{F}(\mathfrak{h}) \\
 & g^* & 
 \end{array} \tag{144}$$

is defined by a degree -1 map  $\eta : \mathfrak{h}^* \oplus \mathfrak{h}^*[1] \rightarrow \text{CE}(\mathfrak{g})$  which is extended to a linear degree -1 map  $\eta : \wedge^\bullet(\mathfrak{h}^* \oplus \mathfrak{h}^*[1]) \rightarrow \text{CE}(\mathfrak{g})$  by defining it on all monomials of generators by the formula

$$\eta : x_1 \wedge \cdots \wedge x_n \mapsto$$

$$\frac{1}{n!} \sum_{\sigma} \epsilon(\sigma) \sum_{k=1}^n (-1)^{\sum_{i=1}^{k-1} |x_{\sigma(i)}|} g^*(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k-1)}) \wedge \eta(x_{\sigma(k)}) \wedge f^*(x_{\sigma(k+1)} \wedge \cdots \wedge x_{\sigma(n)}) \quad (145)$$

for all  $x_1, \dots, x_n \in \mathfrak{h}^* \oplus \mathfrak{h}^*[1]$ , such that this is a chain homotopy from  $f^*$  to  $g^*$ :

$$g^* = f^* + [d, \eta]. \quad (146)$$

- A general 2-morphism

$$\begin{array}{ccc} & f^* & \\ & \curvearrowright & \\ \text{CE}(\mathfrak{g}) & \begin{array}{c} \parallel \\ \eta \end{array} & \text{CE}(\mathfrak{h}) \\ & \curvearrowleft & \\ & g^* & \end{array} \quad (147)$$

is a 2-morphism

$$\begin{array}{ccccc} & & \text{CE}(\mathfrak{h}) & & \\ & g^* \curvearrowright & & i^* \swarrow & \\ \text{CE}(\mathfrak{g}) & & & & W(\mathfrak{h}) \xleftarrow{\cong} F(\mathfrak{h}) \\ & f^* \curvearrowleft & & i^* \swarrow & \\ & & \text{CE}(\mathfrak{h}) & & \end{array} \quad (148)$$

of the above kind that vanishes on the shifted generators, i.e. such that

$$\begin{array}{ccccc} & & \text{CE}(\mathfrak{h}) & & \\ & g^* \curvearrowright & & i^* \swarrow & \\ \text{CE}(\mathfrak{g}) & & & & W(\mathfrak{h}) \xleftarrow{\quad} \mathfrak{h}^*[1] \\ & f^* \curvearrowleft & & i^* \swarrow & \\ & & \text{CE}(\mathfrak{h}) & & \end{array} \quad (149)$$

vanishes.

**Proposition 9** Formula 145 is consistent in that  $g^*|_{\mathfrak{h}^* \oplus \mathfrak{h}^*[1]} = (f^* + [d, \eta])|_{\mathfrak{h}^* \oplus \mathfrak{h}^*[1]}$  implies that  $g^* = f^* + [d, \eta]$  on all elements of  $F(\mathfrak{h})$ .

**Remark.** Definition 17, which may look ad hoc at this point, has a practical and a deep conceptual motivation.

- **Practical motivation.** While it is clear that 2-morphisms of qDGCAs should be chain homotopies, it is not straightforward, in general, to characterize these by their action on generators. Except when the domain qDGCA is free, in which case our formula 17 makes sense. The prescription 148 then provides a systematic algorithm for extending this to arbitrary qDGCAs.

In particular, using the isomorphism  $W(\mathfrak{g}) \simeq F(\mathfrak{g})$  from proposition 6, the above yields the usual explicit description of the homotopy operator  $\tau : W(\mathfrak{g}) \rightarrow W(\mathfrak{g})$  with  $\text{Id}_{W(\mathfrak{g})} = [d_{W(\mathfrak{g})}, \tau]$ . Among other things, this computes for us the transgression elements (“Chern-Simons elements”) for  $L_\infty$ -algebras in 6.3.

- **Conceptual motivation.** As we will see in 6.3 and 6.5, the qDGCA  $W(\mathfrak{g})$  plays an important twofold role: it is both the algebra of differential forms on the total space of the universal  $\mathfrak{g}$ -bundle – while  $\text{CE}(\mathfrak{g})$  is that of forms on the fiber –, as well as the domain for  $\mathfrak{g}$ -valued differential forms, where the shifted component, that in  $\mathfrak{g}^*[1]$ , is the home of the corresponding curvature.

In the light of this, the above restriction 149 can be understood as saying either that

- *vertical* transformations induce transformations on the fibers;

or

- gauge transformations of  $\mathfrak{g}$ -valued forms are transformations under which the curvatures transform covariantly.

**Finite transformations between qDGCA morphisms: concordances.** We now consider the finite transformations of morphisms of DGCA's. What we called 2-morphisms or transformations for qDGCA's above would in other contexts possibly be called a homotopy. Also the following concept is a kind of homotopy, and appears as such in [71] which goes back to [14]. Here we wish to clearly distinguish these different kinds of homotopies and address the following concept as *concordance* – a finite notion of 2-morphism between dg-algebra morphisms.

**Remark.** In the following the algebra of forms  $\Omega^\bullet(I)$  on the interval

$$I := [0, 1]$$

plays an important role. Essentially everything would also go through by instead using  $F(\mathbb{R})$ , the DGCA on a single degree 0 generator, which is the algebra of *polynomial* forms on the interval. This is the model used in [71].

**Definition 18 (concordance)** *We say that two qDGCA morphisms*

$$\text{CE}(\mathfrak{g}) \xleftarrow{g^*} \text{CE}(\mathfrak{h}) \tag{150}$$

*and*

$$\text{CE}(\mathfrak{g}) \xleftarrow{h^*} \text{CE}(\mathfrak{h}) \tag{151}$$

*are concordant, if there exists a dg-algebra homomorphism*

$$\text{CE}(\mathfrak{g}) \otimes \Omega^\bullet(I) \xleftarrow{\eta^*} \text{CE}(\mathfrak{h}) \tag{152}$$

*from the source  $\text{CE}(\mathfrak{h})$  to the target  $\text{CE}(\mathfrak{g})$  tensored with forms on the interval, which restricts to the two given homomorphisms when pulled back along the two boundary inclusions*

$$\{\bullet\} \xrightleftharpoons[t]{s} I, \tag{153}$$

so that the diagram of dg-algebra morphisms

$$\begin{array}{ccc}
 & g^* & \\
 \swarrow & & \searrow \\
 \text{CE}(\mathfrak{g}) & \xrightleftharpoons[\text{Id} \otimes t^*]{\text{Id} \otimes s^*} \text{CE}(\mathfrak{g}) \otimes \Omega^\bullet(I) & \xleftarrow{\eta^*} \text{CE}(\mathfrak{h}) \\
 \nwarrow & & \nearrow \\
 & f^* &
 \end{array} \quad (154)$$

commutes.

See also table 5. Notice that the above diagram is shorthand for two separate commuting diagrams, one involving  $g^*$  and  $s^*$ , the other involving  $f^*$  and  $t^*$ .

We can make precise the statement that definition 17 is the infinitesimal version of definition 18, as follows.

**Proposition 10** *Concordances*

$$\text{CE}(\mathfrak{g}) \otimes \Omega^\bullet(I) \xleftarrow{\eta^*} \text{CE}(\mathfrak{h}) \quad (155)$$

are in bijection with 1-parameter families

$$\alpha : [0, 1] \rightarrow \text{Hom}_{\text{dg-Alg}}(\text{CE}(\mathfrak{h}), \text{CE}(\mathfrak{g})) \quad (156)$$

of morphisms whose derivatives with respect to the parameter is a chain homotopy, i.e. a 2-morphism

$$\begin{array}{ccc}
 & 0 & \\
 \swarrow & & \searrow \\
 \forall t \in [0, 1] : & \text{CE}(\mathfrak{g}) & \xleftrightarrow{\rho} \text{CE}(\mathfrak{h}) \\
 \nwarrow & & \nearrow \\
 & \frac{d}{dt} \alpha(t) = [d, \rho] &
 \end{array} \quad (157)$$

in the 2-category of cochain complexes. For any such  $\alpha$ , the morphisms  $f^*$  and  $g^*$  between which it defines a concordance are defined by the value of  $\alpha$  on the boundary of the interval.

Proof. Writing  $t : [0, 1] \rightarrow \mathbb{R}$  for the canonical coordinate function on the interval  $I = [0, 1]$  we can decompose the dg-algebra homomorphism  $\eta^*$  as

$$\eta^* : \omega \mapsto (t \mapsto \alpha(\omega)(t) + dt \wedge \rho(\omega)(t)). \quad (158)$$

$\alpha$  is itself a degree 0 dg-algebra homomorphism, while  $\rho$  is degree -1 map. Then the fact that  $\eta^*$  respects the differentials implies that for all  $\omega \in \text{CE}(\mathfrak{h})$  we have

$$\begin{array}{ccc}
 \omega & \xrightarrow{d_{\mathfrak{h}}} & d_{\mathfrak{h}} \omega \\
 \downarrow \eta^* & & \downarrow \eta^* \\
 (t \mapsto (\alpha(\omega)(t) + dt \wedge \rho(\omega)(t))) & \xrightarrow{d_{\mathfrak{g}} + d_t} & \begin{aligned} & (t \mapsto (\alpha(d_{\mathfrak{h}} \omega)(t) + dt \wedge \rho(d_{\mathfrak{h}} \omega)(t))) \\ & = (t \mapsto (d_{\mathfrak{g}}(\alpha(\omega))(t) \\ & \quad + dt \wedge (\frac{d}{dt} \alpha(\omega) - d_{\mathfrak{g}} \rho(\omega))(t))) \end{aligned}
 \end{array} \quad (159)$$

The equality in the bottom right corner says that

$$\alpha \circ d_{\mathfrak{h}} - d_{\mathfrak{g}} \circ \alpha = 0 \quad (160)$$

and

$$\forall \omega \in \text{CE}(\mathfrak{g}) : \frac{d}{dt} \alpha(\omega) = \rho(d_{\mathfrak{h}} \omega) + d_{\mathfrak{g}}(\rho(\omega)). \quad (161)$$

But this means that  $\alpha$  is a chain homomorphism whose derivative is given by a chain homotopy.  $\square$

### 6.2.1 Examples

**Transformations between DGCA morphisms.** We demonstrate two examples for the application of the notion of transformations of DGCA morphisms from definition 17 which are relevant for us.

**Computation of transgression forms.** As an example for the transformation in definition 17, we show how the usual Chern-Simons transgression form is computed using formula 145. The reader may wish to first skip to our discussion of Lie  $\infty$ -algebra cohomology in 6.3 for more background. So let  $\mathfrak{g}$  be an ordinary Lie algebra with invariant bilinear form  $P$ , which we regard as a  $d_{W(\mathfrak{g})}$ -closed element  $P \in \wedge^2 \mathfrak{g}^*[1] \subset W(\mathfrak{g})$ . We would like to compute  $\tau P$ , where  $\tau$  is the contracting homotopy of  $W(\mathfrak{g})$ , such that

$$[d, \tau] = \text{Id}_{W(\mathfrak{g})}, \quad (162)$$

which according to proposition 6 is given on generators by

$$\tau : a \mapsto 0 \quad (163)$$

$$\tau : d_{W(\mathfrak{g})} a \mapsto a \quad (164)$$

for all  $a \in \mathfrak{g}^*$ . Let  $\{t^a\}$  be a chosen basis of  $\mathfrak{g}^*$  and let  $\{P_{ab}\}$  be the components of  $P$  in that basis, then

$$P = P_{ab}(\sigma t^a) \wedge (\sigma t^b). \quad (165)$$

In order to apply formula 145 we need to first rewrite this in terms of monomials in  $\{t^a\}$  and  $\{d_{W(\mathfrak{g})} t^a\}$ . Hence, using  $\sigma t^a = d_{W(\mathfrak{g})} t^a + \frac{1}{2} C^a_{bc} t^b \wedge t^c$ , we get

$$\tau P = \tau \left( P_{ab}(d_{W(\mathfrak{g})} t^a) \wedge (d_{W(\mathfrak{g})} t^a) - P_{ab}(d_{W(\mathfrak{g})} t^a) \wedge C^b_{cd} t^c \wedge t^d + \frac{1}{4} P_{ab} C^a_{cd} C^b_{ef} t^c \wedge t^d \wedge t^c \wedge t^d \right). \quad (166)$$

Now equation 145 can be applied to each term. Noticing the combinatorial prefactor  $\frac{1}{n!}$ , which depends on the number of factors in the above terms, and noticing the sum over all permutations, we find

$$\begin{aligned} \tau (P_{ab}(d_{W(\mathfrak{g})} t^a) \wedge (d_{W(\mathfrak{g})} t^a)) &= P_{ab}(d_{W(\mathfrak{g})} t^a) \wedge t^b \\ \tau (-P_{ab}(d_{W(\mathfrak{g})} t^a) \wedge C^b_{cd} t^c \wedge t^d) &= \frac{1}{3!} \cdot 2 P_{ab} C^b_{cd} t^b \wedge t^c \wedge t^d = \frac{1}{3} C_{abc} t^a \wedge t^b \wedge t^c, \end{aligned} \quad (167)$$

where we write  $C_{abc} := P_{ad} C^d_{bc}$  as usual. Finally  $\tau \left( \frac{1}{4} P_{ab} C^a_{cd} C^b_{ef} t^c \wedge t^d \wedge t^c \wedge t^d \right) = 0$ . In total this yields

$$\tau P = P_{ab}(d_{W(\mathfrak{g})} t^a) \wedge t^b + \frac{1}{3} C_{abc} t^a \wedge t^b \wedge t^c. \quad (168)$$

By again using  $d_{W(\mathfrak{g})} t^a = -\frac{1}{2} C^a_{bc} t^b \wedge t^c + \sigma t^a$  together with the invariance of  $P$  (hence the  $d_{W(\mathfrak{g})}$ -closedness of  $P$  which implies that the constants  $C_{abc}$  are skew symmetric in all three indices), one checks that this does indeed satisfy

$$d_{W(\mathfrak{g})} \tau P = P. \quad (169)$$

In 6.5 we will see that after choosing a  $\mathfrak{g}$ -valued connection on the space  $Y$  the generators  $t^a$  here will get sent to components of a  $\mathfrak{g}$ -valued 1-form  $A$ , while the  $d_{W(\mathfrak{g})}t^a$  will get sent to the components of  $dA$ . Under this map the element  $\tau P \in W(\mathfrak{g})$  maps to the familiar Chern-Simons 3-form

$$CS_P(A) := P(A \wedge dA) + \frac{1}{3}P(A \wedge [A \wedge A]) \quad (170)$$

whose differential is the characteristic form of  $A$  with respect to  $P$ :

$$dCS_P(A) = P(F_A \wedge F_A). \quad (171)$$

Characteristic forms, for arbitrary Lie  $\infty$ -algebra valued forms, are discussed further in 6.6.

## 2-Morphisms of Lie 2-algebras

**Proposition 11** *For the special case that  $\mathfrak{g}$  is any Lie 2-algebra (any  $L_\infty$ -algebra concentrated in the first two degrees) the 2-morphisms defined by definition 17 reproduce the 2-morphisms of Lie 2-algebras as stated in [5] and used in [7].*

Proof. The proof is given in the appendix.  $\square$

This implies in particular that with the 1- and 2-morphisms as defined above, Lie 2-algebras do form a 2-category. There is an rather straightforward generalization of definition 17 to higher morphisms, which one would expect yields correspondingly  $n$ -categories of Lie  $n$ -algebras. But this we shall not try to discuss here.

## 6.3 $L_\infty$ -algebra cohomology

The study of ordinary Lie algebra cohomology and of invariant polynomials on the Lie algebra has a simple formulation in terms of the qDGCAs  $CE(\mathfrak{g})$  and  $W(\mathfrak{g})$ . Furthermore, this has a straightforward generalization to arbitrary  $L_\infty$ -algebras which we now state.

For  $CE(\mathfrak{g}) \xleftarrow{i^*} W(\mathfrak{g})$  the canonical morphism from proposition 6, notice that

$$CE(\mathfrak{g}) \simeq W(\mathfrak{g})/\ker(i^*) \quad (172)$$

and that

$$\ker(i^*) = \langle \mathfrak{g}^*[1] \rangle_{W(\mathfrak{g})}, \quad (173)$$

the ideal in  $W(\mathfrak{g})$  generated by  $\mathfrak{g}^*[1]$ . Algebra derivations

$$\iota_X : W(\mathfrak{g}) \rightarrow W(\mathfrak{g}) \quad (174)$$

for  $X \in \mathfrak{g}$  are like (contractions with) vector fields on the space on which  $W(\mathfrak{g})$  is like differential forms. In the case of an ordinary Lie algebra  $\mathfrak{g}$ , the corresponding inner derivations  $[d_{W(\mathfrak{g})}, \iota_X]$  for  $X \in \mathfrak{g}$  are of degree -1 and are known as the Lie derivative  $L_X$ . They generate flows  $\exp([d_{W(\mathfrak{g})}, \iota_X]) : W(\mathfrak{g}) \rightarrow W(\mathfrak{g})$  along these vector fields.

**Definition 19 (vertical derivations)** *We say an algebra derivation  $\tau : W(\mathfrak{g}) \rightarrow W(\mathfrak{g})$  is vertical if it vanishes on the shifted copy  $\mathfrak{g}^*[1]$  of  $\mathfrak{g}^*$  inside  $W(\mathfrak{g})$ ,*

$$\tau|_{\mathfrak{g}^*[1]} = 0. \quad (175)$$

**Proposition 12** *The vertical derivations are precisely those that come from contractions*

$$\iota_X : \mathfrak{g}^* \mapsto \mathbb{R} \quad (176)$$

for all  $X \in \mathfrak{g}$ , extended to 0 on  $\mathfrak{g}^*[1]$  and extended as algebra derivations to all of  $\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])$ .

The reader should compare this and the following definitions to the theory of vertical Lie derivatives and basic differential forms with respect to any surjective submersion  $\pi : Y \rightarrow X$ . This is discussed in 5.3.1.

**Definition 20 (basic forms and invariant polynomials)** *The algebra  $W(\mathfrak{g})_{\text{basic}}$  of basic forms in  $W(\mathfrak{g})$  is the intersection of the kernels of all vertical derivations and Lie derivatives. i.e. all the contractions  $\iota_X$  and Lie derivatives  $L_X$  for  $X \in \mathfrak{g}$ . Since  $L_X = [d_{W(\mathfrak{g})}, \iota_X]$ , it follows that in the kernel of  $\iota_X$ , the Lie derivative vanishes only if  $\iota_X d_{W(\mathfrak{g})}$  vanishes.*

As will be discussed in a moment, basic forms in  $W(\mathfrak{g})$  play the role of **invariant polynomials** on the  $L_\infty$ -algebra  $\mathfrak{g}$ . Therefore we often write  $\text{inv}(\mathfrak{g})$  for  $W(\mathfrak{g})_{\text{basic}}$ :

$$\text{inv}(\mathfrak{g}) := W(\mathfrak{g})_{\text{basic}} . \quad (177)$$

Using the obvious inclusion  $W(\mathfrak{g}) \xleftarrow{p^*} \text{inv}(\mathfrak{g})$  we obtain the sequence

$$\text{CE}(\mathfrak{g}) \xleftarrow{i^*} W(\mathfrak{g}) \xleftarrow{p^*} \text{inv}(\mathfrak{g}) \quad (178)$$

of dg-algebras that plays a major role in our analysis: it can be interpreted as coming from the universal bundle for the Lie  $\infty$ -algebra  $\mathfrak{g}$ . As shown in figure 4, we can regard vertical derivations on  $W(\mathfrak{g})$  as derivations along the *fibers* of the corresponding dual sequence.

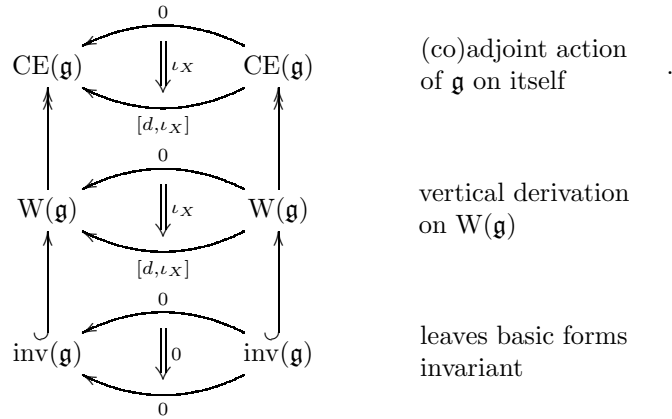


Figure 4: **Interpretation of vertical derivations on  $W(\mathfrak{g})$ .** The algebra  $\text{CE}(\mathfrak{g})$  plays the role of the algebra of differential forms on the Lie  $\infty$ -group that integrates the Lie  $\infty$ -algebra  $\mathfrak{g}$ . The coadjoint action of  $\mathfrak{g}$  on these forms corresponds to Lie derivatives along the fibers of the universal bundle. These vertical derivatives leave the forms on the base of this universal bundle invariant. The diagram displayed is in the 2-category  $\mathbf{Ch}^\bullet$  of cochain complexes, as described in the beginning of 6.2.

**Definition 21 (cocycles, invariant polynomials and transgression elements)** *Let  $\mathfrak{g}$  be an  $L_\infty$ -algebra. Then*

- An  $L_\infty$ -algebra **cocycle** on  $\mathfrak{g}$  is a  $d_{\text{CE}(\mathfrak{g})}$ -closed element of  $\text{CE}(\mathfrak{g})$ .

$$\mu \in \text{CE}(\mathfrak{g}) , \quad d_{\text{CE}(\mathfrak{g})}\mu = 0 . \quad (179)$$

- An  $L_\infty$ -algebra **invariant polynomial** on  $\mathfrak{g}$  is an element  $P \in \text{inv}(\mathfrak{g}) := W(\mathfrak{g})_{\text{basic}}$ .

- An  $L_\infty$ -algebra  **$\mathfrak{g}$ -transgression element** for a given cocycle  $\mu$  and an invariant polynomial  $P$  is an element  $cs \in W(\mathfrak{g})$  such that

$$d_{W(\mathfrak{g})}cs = p^*P \quad (180)$$

$$i^*cs = \mu. \quad (181)$$

If a transgression element for  $\mu$  and  $P$  exists, we say that  $\mu$  **transgresses to  $P$**  and that  $P$  **suspends to  $\mu$** . If  $\mu = 0$  we say that  $P$  **suspends to 0**. The situation is illustrated diagrammatically in figure 5 and figure 6.

**Definition 22 (suspension to 0)** An element  $P \in \text{inv}(\mathfrak{g})$  is said to suspend to 0 if under the inclusion

$$\ker(i^*) \xleftarrow{p^*} W(\mathfrak{g}) \quad (182)$$

it becomes a coboundary:

$$p^*P = d_{\ker(i^*)}\alpha \quad (183)$$

for some  $\alpha \in \ker(i^*)$ .

**Remark.** We will see that it is the intersection of  $\text{inv}(\mathfrak{g})$  with the cohomology of  $\ker(i^*)$  that is a candidate, in general, for an algebraic model of the classifying space of the object that integrates the  $L_\infty$ -algebra  $\mathfrak{g}$ . But at the moment we do not entirely clarify this relation to the integrated theory, unfortunately.

cocycle                  transgression element                  inv. polynomial

$$\begin{array}{ccccc}
 G & \xrightarrow{i} & EG & \xrightarrow{p} & BG \\
 & & & & \\
 & & 0 & & \\
 & & \uparrow d & & \\
 & & p^*P & \xleftarrow{p^*} & P \\
 & & \uparrow d & & \\
 0 & & & & \\
 \uparrow d & & & & \\
 \mu & \xleftarrow{i^*} & cs & & 
 \end{array}$$

Figure 5: **Lie algebra cocycles, invariant polynomials and transgression forms** in terms of cohomology of the universal  $G$ -bundle. Let  $G$  be a simply connected compact Lie group with Lie algebra  $\mathfrak{g}$ . Then invariant polynomials  $P$  on  $\mathfrak{g}$  correspond to elements in the cohomology  $H^\bullet(BG)$  of the classifying space of  $G$ . When pulled back to the total space of the universal  $G$ -bundle  $EG \rightarrow BG$ , these classes become trivial, due to the contractability of  $EG$ :  $p^*P = d(cs)$ . Lie algebra cocycles, on the other hand, correspond to elements in the cohomology  $H^\bullet(G)$  of  $G$  itself. A cocycle  $\mu \in H^\bullet(G)$  is in transgression with an invariant polynomial  $P \in H^\bullet(BG)$  if  $\mu = i^*cs$ .

**Proposition 13** For the case that  $\mathfrak{g}$  is an ordinary Lie algebra, the above definition reproduces the ordinary definitions of Lie algebra cocycles, invariant polynomials, and transgression elements. Moreover, all elements in  $\text{inv}(\mathfrak{g})$  are closed.



cocycle                      transgression element                      inv. polynomial

$$\begin{array}{c}
 \text{CE}(\mathfrak{g}) \xleftarrow{i^*} W(\mathfrak{g}) \xleftarrow{p^*} \text{inv}(\mathfrak{g}) \\
 \\
 \begin{array}{ccc}
 & 0 & \\
 & \uparrow d_{W(\mathfrak{g})} & \\
 0 & p^* P & \xleftarrow{p^*} P \\
 \uparrow d_{\text{CE}(\mathfrak{g})} & \downarrow d_{W(\mathfrak{g})} & \\
 \mu & \xleftarrow{i^*} \text{CS} & 
 \end{array}
 \end{array}$$

Figure 6: **The homotopy operator**  $\tau$  is a contraction homotopy for  $W(\mathfrak{g})$ . Acting with it on a closed invariant polynomial  $P \in \text{inv}(\mathfrak{g}) \subset \wedge^\bullet \mathfrak{g}[1] \subset W(\mathfrak{g})$  produces an element  $\text{cs} \in W(\mathfrak{g})$  whose “restriction to the fiber”  $\mu := i^* \text{cs}$  is necessarily closed and hence a cocycle. We say that  $\text{cs}$  induces the *transgression* from  $\mu$  to  $P$ , or that  $P$  *suspends* to  $\mu$ .

Proof. That the definitions of Lie algebra cocycles and transgression elements coincides is clear. It remains to be checked that  $\text{inv}(\mathfrak{g})$  really contains the invariant polynomials. In the ordinary definition a  $\mathfrak{g}$ -invariant polynomial is a  $d_{W(\mathfrak{g})}$ -closed element in  $\wedge^\bullet(\mathfrak{g}^*[1])$ . Hence one only needs to check that all elements in  $\wedge^\bullet(\mathfrak{g}^*[1])$  with the property that their image under  $d_{W(\mathfrak{g})}$  is again in  $\wedge^\bullet(\mathfrak{g}^*[1])$  are in fact already closed. This can be seen for instance in components, using the description of  $W(\mathfrak{g})$  given in 6.1.1.  $\square$

**Remark.** For ordinary Lie algebras  $\mathfrak{g}$  corresponding to a simply connected compact Lie group  $G$ , the situation is often discussed in terms of the cohomology of the universal  $G$ -bundle. This is recalled in figure 5 and in 6.3.1. The general definition above is a precise analog of that familiar situation:  $W(\mathfrak{g})$  plays the role of the algebra of (left invariant) differential forms on the universal  $\mathfrak{g}$ -bundle and  $\text{CE}(\mathfrak{g})$  plays the role of the algebra of (left invariant) differential forms on its fiber. Then  $\text{inv}(\mathfrak{g})$  plays the role of differential forms on the base,  $BG = EG/G$ . In fact, for  $G$  a compact and simply connected Lie group and  $\mathfrak{g}$  its Lie algebra, we have

$$H^\bullet(\text{inv}(\mathfrak{g})) \simeq H^\bullet(BG, \mathbb{R}). \quad (184)$$

In summary, the situation we thus obtain is that depicted in figure 1. Compare this to the following fact.

**Proposition 14** *For  $p : P \rightarrow X$  a principal  $G$ -bundle, let  $\text{vert}(P) \subset \Gamma(TP)$  be the vertical vector fields on  $P$ . The horizontal differential forms on  $P$  which are invariant under  $\text{vert}(P)$  are precisely those that are pulled back along  $p$  from  $X$ .*

These are called the **basic differential forms** in [41].

**Remark.** We will see that, contrary to the situation for ordinary Lie algebras, in general invariant polynomials of  $L_\infty$  algebras are not  $d_{W(\mathfrak{g})}$ -closed (the  $d_{W(\mathfrak{g})}$ -differential of them is just horizontal). We will also see that those indecomposable invariant polynomials in  $\text{inv}(\mathfrak{g})$ , i.e. those that become exact in  $\ker(i^*)$ , are not characteristic for the corresponding  $\mathfrak{g}$ -bundles. This probably means that the real cohomology of the classifying space of the Lie  $\infty$ -group integrating  $\mathfrak{g}$  is spanned by invariant polynomials modulo those suspending to 0. But here we do not attempt to discuss this further.

**Proposition 15** *For every invariant polynomial  $P \in \wedge^\bullet \mathfrak{g}[1] \subset W(\mathfrak{g})$  on an  $L_\infty$ -algebra  $\mathfrak{g}$  such that  $d_{W(\mathfrak{g})}p^*P = 0$ , there exists an  $L_\infty$ -algebra cocycle  $\mu \in CS(\mathfrak{g})$  that transgresses to  $P$ .*

Proof. This is a consequence of proposition 6 and proposition 1. Let  $P \in W(\mathfrak{g})$  be an invariant polynomial. By proposition 6,  $p^*P$  is in the kernel of the restriction homomorphism  $CE(\mathfrak{g}) \xleftarrow{i^*} W(\mathfrak{g}) : i^*P = 0$ . By proposition 1,  $p^*P$  is the image under  $d_{W(\mathfrak{g})}$  of an element  $cs := \tau(p^*P)$  and by the algebra homomorphism property of  $i^*$  we know that its restriction,  $\mu := i^*cs$ , to the fiber is closed, because

$$d_{CE(\mathfrak{g})}i^*cs = i^*d_{W(\mathfrak{g})}cs = i^*p^*P = 0. \quad (185)$$

Therefore  $\mu$  is an  $L_\infty$ -algebra cocycle for  $\mathfrak{g}$  that transgresses to the invariant polynomial  $P$ .  $\square$

**Remark.** Notice that this statement is useful only for *indecomposable* invariant polynomials. All others trivially suspend to the 0 cocycle.

**Proposition 16** *An invariant polynomial which suspends to a Lie  $\infty$ -algebra cocycle that is a coboundary also suspends to 0.*

Proof. Let  $P$  be an invariant polynomial,  $cs$  the corresponding transgression element and  $\mu = i^*cs$  the corresponding cocycle, which is assumed to be a coboundary in that  $\mu = d_{CE(\mathfrak{g})}b$  for some  $b \in CE(\mathfrak{g})$ . Then by the definition of  $d_{W(\mathfrak{g})}$  it follows that  $\mu = i^*(d_{W(\mathfrak{g})}b)$ .

Now notice that

$$cs' := cs - d_{W(\mathfrak{g})}b \quad (186)$$

is another transgression element for  $P$ , since

$$d_{W(\mathfrak{g})}cs' = p^*P. \quad (187)$$

But now

$$i^*(cs') = i^*(cs - d_{W(\mathfrak{g})}b) = 0. \quad (188)$$

Hence  $P$  suspends to 0.  $\square$

### 6.3.1 Examples

**The cohomologies of  $G$  and of  $BG$  in terms of qDGCA's.** To put our general considerations for  $L_\infty$ -algebras into perspective, it is useful to keep the following classical results for ordinary Lie algebras in mind.

A classical result of E. Cartan [22] [23] (see also [49]) says that for a connected finite dimensional Lie group  $G$ , the cohomology  $H^\bullet(G)$  of the group is isomorphic to that of the Chevalley-Eilenberg algebra  $CE(\mathfrak{g})$  of its Lie algebra  $\mathfrak{g}$ :

$$H^\bullet(G) \simeq H^\bullet(CE(\mathfrak{g})), \quad (189)$$

namely to the algebra of Lie algebra cocycles on  $\mathfrak{g}$ . If we denote by  $Q_G$  the space of *indecomposable* such cocycles, and form the qDGCA  $\wedge^\bullet Q_G = H^\bullet(\wedge^\bullet Q_G)$  with trivial differential, the above says that we have an isomorphism in cohomology

$$H^\bullet(G) \simeq H^\bullet(\wedge^\bullet Q_G) = \wedge^\bullet Q_G \quad (190)$$

which is realized by the canonical inclusion

$$i : \wedge^\bullet Q_G \hookrightarrow CE(\mathfrak{g}) \quad (191)$$

of all cocycles into the Chevalley-Eilenberg algebra.

Subsequently, we have the classical result of Borel [13]: For a connected finite dimensional Lie group  $G$ , the cohomology of its classifying space  $BG$  is a finitely generated polynomial algebra on even dimensional generators:

$$H^\bullet(BG) \simeq \wedge^\bullet P_G. \quad (192)$$

Here  $P_G$  is the space of *indecomposable* invariant polynomials on  $\mathfrak{g}$ , hence

$$H^\bullet(BG) \simeq H^\bullet(\text{inv}(\mathfrak{g})). \quad (193)$$

In fact,  $P_G$  and  $Q_G$  are isomorphic after a shift:

$$P_G \simeq Q_G[1] \quad (194)$$

and this isomorphism is induced by *transgression* between indecomposable cocycles  $\mu \in \text{CE}(\mathfrak{g})$  and indecomposable invariant polynomials  $P \in \text{inv}(\mathfrak{g})$  via a transgression element  $\text{cs} = \tau P \in W(\mathfrak{g})$ .

### Cohomology and invariant polynomials of $b^{n-1}\mathfrak{u}(1)$

**Proposition 17** *For every integer  $n \geq 1$ , the Lie  $n$ -algebra  $b^{n-1}\mathfrak{u}(1)$  (the  $(n-1)$ -folded shifted version of ordinary  $\mathfrak{u}(1)$ ) from 6.1.1) we have the following:*

- there is, up to a scalar multiple, a single indecomposable Lie  $\infty$ -algebra cocycle which is of degree  $n$  and linear,

$$\mu_{b^{n-1}\mathfrak{u}(1)} \in \mathbb{R}[n] \subset \text{CE}(b^{n-1}\mathfrak{u}(1)), \quad (195)$$

- there is, up to a scalar multiple, a single indecomposable Lie  $\infty$ -algebra invariant polynomial, which is of degree  $(n+1)$

$$P_{b^{n-1}\mathfrak{u}(1)} \in \mathbb{R}[n+1] \subset \text{inv}(b^{n-1}\mathfrak{u}(1)) = \text{CE}(b^n\mathfrak{u}(1)). \quad (196)$$

- The cocycle  $\mu_{b^{n-1}\mathfrak{u}(1)}$  is in transgression with  $P_{b^{n-1}\mathfrak{u}(1)}$ .

These statements are an obvious consequence of the definitions involved, but they are important. The fact that  $b^{n-1}\mathfrak{u}(1)$  has a single invariant polynomial of degree  $(n+1)$  will immediately imply, in 7, that  $b^{n-1}\mathfrak{u}(1)$ -bundles have a single characteristic class of degree  $(n+1)$ : known (at least for  $n=2$ , as the Dixmier-Douady class). Such a  $b^{n-1}\mathfrak{u}(1)$ -bundle classes appear in 8 as the obstruction classes for lifts of  $n$ -bundles through string-like extensions of their structure Lie  $n$ -algebra.

**Cohomology and invariant polynomials of strict Lie 2-algebras.** Let  $\mathfrak{g}_{(2)} = (\mathfrak{h} \xrightarrow{t} \mathfrak{g} \xrightarrow{\alpha} \text{der}(\mathfrak{h}))$  be a strict Lie 2-algebra as described in section 6.1. Notice that there is a canonical projection homomorphism

$$\text{CE}(\mathfrak{g}) \xleftarrow{j^*} \text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \quad (197)$$

which, of course, extends to the Weil algebras

$$W(\mathfrak{g}) \xleftarrow{j^*} W(\mathfrak{h} \xrightarrow{t} \mathfrak{g}). \quad (198)$$

Here  $j^*$  is simply the identity on  $\mathfrak{g}^*$  and on  $\mathfrak{g}^*[1]$  and vanishes on  $\mathfrak{h}^*[1]$  and  $\mathfrak{h}^*[2]$ .

**Proposition 18** *Every invariant polynomial  $P \in \text{inv}(\mathfrak{g})$  of the ordinary Lie algebra  $\mathfrak{g}$  lifts to an invariant polynomial on the Lie 2-algebra  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ :*

$$\begin{array}{ccc} W(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) & & \\ \downarrow i^* & \swarrow & \\ W(\mathfrak{g}) & \xleftarrow{\quad} & \text{inv}(\mathfrak{g}) \end{array} \quad (199)$$

However, a closed invariant polynomial will not necessarily lift to a closed one.

Proof. Recall that  $d_t := d_{\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}$  acts on  $\mathfrak{g}^*$  as

$$d_t|_{\mathfrak{g}^*} = [\cdot, \cdot]_{\mathfrak{g}}^* + t^*. \quad (200)$$

By definition 15 and definition 16 it follows that  $d_{\text{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}$  acts on  $\mathfrak{g}^*[1]$  as

$$d_{\text{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}|_{\mathfrak{g}^*[1]} = -\sigma \circ [\cdot, \cdot]_{\mathfrak{g}}^* - \sigma \circ t^* \quad (201)$$

and on  $\mathfrak{h}^*[1]$  as

$$d_{\text{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}|_{\mathfrak{h}^*[1]} = -\sigma \circ \alpha^*. \quad (202)$$

Then notice that

$$(\sigma \circ t^*) : \mathfrak{g}^*[1] \rightarrow \mathfrak{h}^*[2]. \quad (203)$$

But this means that  $d_{\text{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}$  differs from  $d_{\text{W}(\mathfrak{g})}$  on  $\wedge^\bullet(\mathfrak{g}^*[1])$  only by elements that are annihilated by vertical  $\iota_X$ . This proves the claim.  $\square$

It may be easier to appreciate this proof by looking at what it does in terms of a chosen basis.

**Same discussion in terms of a basis.** Let  $\{t^a\}$  be a basis of  $\mathfrak{g}^*$  and  $\{b^i\}$  be a basis of  $\mathfrak{h}^*[1]$ . Let  $\{C^a_{bc}\}$ ,  $\{\alpha^i_{aj}\}$ , and  $\{t^a_i\}$ , respectively, be the components of  $[\cdot, \cdot]_{\mathfrak{g}}$ ,  $\alpha$  and  $t$  in that basis. Then corresponding to  $\text{CE}(\mathfrak{g})$ ,  $\text{W}(\mathfrak{g})$ ,  $\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ , and  $\text{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ , respectively, we have the differentials

$$d_{\text{CE}(\mathfrak{g})} : t^a \mapsto -\frac{1}{2}C^a_{bc}t^b \wedge t^c, \quad (204)$$

$$d_{\text{W}(\mathfrak{g})} : t^a \mapsto -\frac{1}{2}C^a_{bc}t^b \wedge t^c + \sigma t^a, \quad (205)$$

$$d_{\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} : t^a \mapsto -\frac{1}{2}C^a_{bc}t^b \wedge t^c + t^a_i b^i, \quad (206)$$

and

$$d_{\text{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} : t^a \mapsto -\frac{1}{2}C^a_{bc}t^b \wedge t^c + t^a_i b^i + \sigma t^a. \quad (207)$$

Hence we get

$$d_{\text{W}(\mathfrak{g})} : \sigma t^a \mapsto -\sigma(-\frac{1}{2}C^a_{bc}t^b \wedge t^c) = C^a_{bc}(\sigma t^b) \wedge t^c \quad (208)$$

as well as

$$d_{\text{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} : \sigma t^a \mapsto -\sigma(-\frac{1}{2}C^a_{bc}t^b \wedge t^c + t^a_i b^i) = C^a_{bc}(\sigma t^b) \wedge t^c + t^a_i \sigma b^i. \quad (209)$$

Then if

$$P = P_{a_1 \dots a_n}(\sigma t^{a_1}) \wedge \dots \wedge (\sigma t^{a_n}) \quad (210)$$

is  $d_{\text{W}(\mathfrak{g})}$ -closed, i.e. an invariant polynomial on  $\mathfrak{g}$ , it follows that

$$d_{\text{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} P = n P_{a_1, a_2, \dots, a_n}(t^{a_1}_i \sigma b^i) \wedge (\sigma t^{a_2}) \wedge \dots \wedge (\sigma t^{a_n}). \quad (211)$$

(all terms appearing are in the image of the shifting isomorphism  $\sigma$ ), hence  $P$  is also an invariant polynomial on  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ .  $\square$

We will see a physical application of this fact in 6.6.

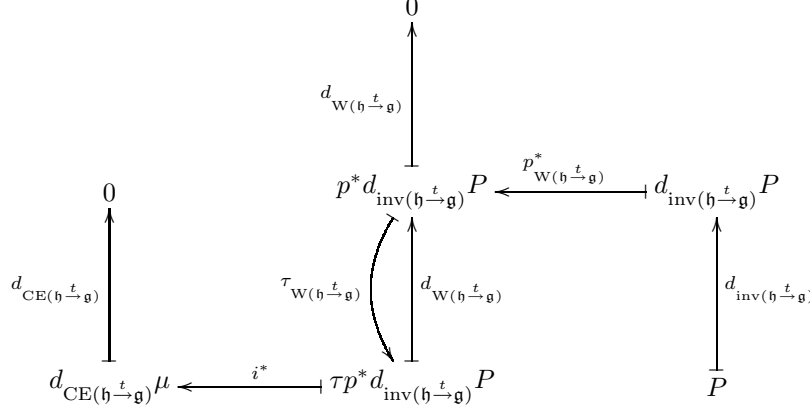


Figure 7: **Cocycles and invariant polynomials on strict Lie 2-algebras  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$** , induced from cocycles and invariant polynomials on  $\mathfrak{g}$ . An invariant polynomial  $P$  on  $\mathfrak{g}$  in transgression with a cocycle  $\mu$  on  $\mathfrak{g}$  lifts to a generally non-closed invariant polynomial on  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ . The diagram says that its closure,  $d_{\text{inv}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} P$ , suspends to the  $d_{\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}$ -closure of the cocycle  $\mu$ . Since this  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ -cocycle  $d_{(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} \mu$  is hence a coboundary, it follows from proposition 16 that  $d_{\text{inv}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} P$  suspends also to  $0$ . Nevertheless the situation is of interest, in that it governs the topological field theory known as BF theory. This is discussed in section 6.6.1.

**Remark.** Notice that the invariant polynomials  $P$  lifted from  $\mathfrak{g}$  to  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  this way are no longer *closed*, in general. This is a new phenomenon we encounter for higher  $L_\infty$ -algebras. While, according to proposition 13, for  $\mathfrak{g}$  an ordinary Lie algebra all elements in  $\text{inv}(\mathfrak{g})$  are closed, this is no longer the case here: the lifted elements  $P$  above vanish only after we hit with them with both  $d_{W(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}$  and a vertical  $\tau$ .

**Proposition 19** *Let  $P$  be any invariant polynomial on the ordinary Lie algebra  $\mathfrak{g}$  in transgression with the cocycle  $\mu$  on  $\mathfrak{g}$ . Regarded both as elements of  $W(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  and  $\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  respectively. Notice that  $d_{\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} \mu$  is in general non-vanishing but is of course now an exact cocycle on  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ .*

*We have : the  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ -cocycle  $d_{\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} \mu$  transgresses to  $d_{\text{inv}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} P$ .*

The situation is illustrated by the diagram in figure 7.

**Concrete Example:  $\mathfrak{su}(5) \rightarrow \mathfrak{sp}(5)$ .** It is known that the cohomology of the Chevalley-Eilenberg algebras for  $\mathfrak{su}(5)$  and  $\mathfrak{sp}(5)$  are generated, respectively, by four and five indecomposable cocycles,

$$H^\bullet(\text{CE}(\mathfrak{su}(5))) = \wedge^\bullet[a, b, c, d] \quad (212)$$

and

$$H^\bullet(\text{CE}(\mathfrak{sp}(5))) = \wedge^\bullet[v, w, x, y, z], \quad (213)$$

	<b>generator</b>	<b>degree</b>
$H^\bullet \text{CE}(\mathfrak{su}(5))$	$a$	3
	$b$	5
	$c$	7
	$d$	9
which have degree as indicated in the following table:		
$H^\bullet(\text{CE}(\mathfrak{sp}(5)))$	$v$	3
	$w$	7
	$x$	11
	$y$	15
	$z$	19

As discussed for instance in [41], the inclusion of groups

$$\text{SU}(5) \hookrightarrow \text{Sp}(5) \quad (214)$$

is reflected in the morphism of DGCAs

$$\text{CE}(\mathfrak{su}(5)) \xleftarrow{t^*} \text{CE}(\mathfrak{sp}(5)) \quad (215)$$

which acts, in cohomology, on  $v$  and  $w$  as

$$\begin{aligned} a &\longleftarrow v \\ c &\longleftarrow w \end{aligned} \quad (216)$$

and which sends  $x$ ,  $y$  and  $z$  to wedge products of generators.

We would like to apply the above reasoning to this situation. Now,  $\mathfrak{su}(5)$  is not normal in  $\mathfrak{sp}(5)$  hence  $(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))$  does not give a Lie 2-algebra. But we can regard the cohomology complexes  $H^\bullet(\text{CE}(\mathfrak{su}(5)))$  and  $H^\bullet(\text{CE}(\mathfrak{sp}(5)))$  as Chevalley-Eilenberg algebras of abelian  $L_\infty$ -algebras in their own right. Their inclusion is normal, in the sense to be made precise below in definition 38. By useful abuse of notation, we write now  $\text{CE}(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))$  for this inclusion at the level of cohomology.

Recalling from 129 that this means that in  $\text{CE}(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))$  we have

$$d_{\text{CE}(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))} v := \sigma a \quad (217)$$

and

$$d_{\text{CE}(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))} w := \sigma c \quad (218)$$

we see that the generators  $\sigma a$  and  $\sigma b$  drop out of the cohomology of the Chevalley-Eilenberg algebra

$$\text{CE}(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5)) = (\bigwedge^\bullet(\mathfrak{sp}(5)^* \oplus \mathfrak{su}(5)^*[1]), d_t) \quad (219)$$

of the strict Lie 2-algebra coming from the infinitesimal crossed module  $(t : \mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))$ .

A simple spectral sequence argument shows that products are not killed in  $H^\bullet(\text{CE}(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5)))$ , but they may no longer be decomposable. Hence

$$H^\bullet(\text{CE}(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))) \quad (220)$$

is generated by classes in degrees 6 and 10 by  $\sigma b$  and  $\sigma d$ , and in degrees 21 and 25, which are represented by products in 219 involving  $\sigma a$  and  $\sigma c$ , with the only non zero product being

$$6 \wedge 25 = 10 \wedge 21, \quad (221)$$

where 31 is the dimension of the manifold  $\text{Sp}(5)/\text{SU}(5)$ . Thus the strict Lie 2-algebra  $(t : \mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))$  plays the role of the quotient Lie 1-algebra  $\mathfrak{sp}(5)/\mathfrak{su}(5)$ . We will discuss the general mechanism behind this phenomenon in 8.1: the Lie 2-algebra  $(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))$  is the *weak cokernel*, i.e. the *homotopy cokernel* of the inclusion  $\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5)$ .

The Weil algebra of  $(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))$  is

$$W(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5)) = (\wedge^\bullet(\mathfrak{sp}(5)^* \oplus \mathfrak{su}(5)^*[1] \oplus \mathfrak{sp}(5)^*[1] \oplus \mathfrak{su}(5)^*[2]), d_{W(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))}). \quad (222)$$

Recall the formula 124 for the action of  $d_{W(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))}$  on generators in  $\mathfrak{sp}(5)^*[1] \oplus \mathfrak{su}(5)^*[2]$ . By that formula,  $\sigma v$  and  $\sigma w$  are invariant polynomials on  $\mathfrak{sp}(5)$  which lift to non-closed invariant polynomials on  $\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5)$ :

$$d_{W(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))} : \sigma v \mapsto -\sigma(d_{\text{CE}(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))} v) = -\sigma \sigma a \quad (223)$$

by equation 217; and

$$d_{W(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))} : \sigma w \mapsto -\sigma(d_{\text{CE}(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))} w) = -\sigma \sigma c \quad (224)$$

by equation 218. Hence  $\sigma v$  and  $\sigma w$  are not closed in  $\text{CE}(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))$ , but they are still invariant polynomials according to definition 20, since their differential sits entirely in the shifted copy  $(\mathfrak{sp}(5)^* \oplus \mathfrak{su}(5)^*[1])[1]$ .

On the other hand, notice that we do also have closed invariant polynomials on  $(\mathfrak{su}(5) \hookrightarrow \mathfrak{sp}(5))$ , for instance  $\sigma \sigma b$  and  $\sigma \sigma d$ .

## 6.4 $L_\infty$ -algebras from cocycles: String-like extensions

We now consider the main object of interest here: families of  $L_\infty$ -algebras that are induced from  $L_\infty$ -cocycles and invariant polynomials. First we need the following

**Definition 23 (String-like extensions of  $L_\infty$ -algebras)** *Let  $\mathfrak{g}$  be an  $L_\infty$ -algebra.*

- For each degree  $(n+1)$ -cocycle  $\mu$  on  $\mathfrak{g}$ , let  $\mathfrak{g}_\mu$  be the  $L_\infty$ -algebra defined by

$$\text{CE}(\mathfrak{g}_\mu) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathbb{R}[n]), d_{\text{CE}(\mathfrak{g}_\mu)}) \quad (225)$$

with differential given by

$$d_{\text{CE}(\mathfrak{g}_\mu)}|_{\mathfrak{g}^*} := d_{\text{CE}(\mathfrak{g})}, \quad (226)$$

and

$$d_{\text{CE}(\mathfrak{g}_\mu)}|_{\mathbb{R}[n]} : b \mapsto -\mu, \quad (227)$$

where  $\{b\}$  denotes the canonical basis of  $\mathbb{R}[n]$ . This we call the **String-like extension** of  $\mathfrak{g}$  with respect to  $\mu$ , because, as described below in 6.4.1, it generalizes the construction of the String Lie 2-algebra.

- For each degree  $n$  invariant polynomial  $P$  on  $\mathfrak{g}$ , let  $\text{ch}_P(\mathfrak{g})$  be the  $L_\infty$ -algebra defined by

$$\text{CE}(\text{ch}_P(\mathfrak{g})) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \oplus \mathbb{R}[2n-1]), d_{\text{CE}(\text{ch}_P(\mathfrak{g}))}) \quad (228)$$

with the differential given by

$$d_{\text{CE}(\text{ch}_P(\mathfrak{g}))}|_{\mathfrak{g}^* \oplus \mathfrak{g}^*[1]} := d_{W(\mathfrak{g})} \quad (229)$$

and

$$d_{\text{CE}(\text{ch}_P(\mathfrak{g}))}|_{\mathbb{R}[2n-1]} : c \mapsto P, \quad (230)$$

where  $\{c\}$  denotes the canonical basis of  $\mathbb{R}[2n-1]$ . This we call the **Chern  $L_\infty$ -algebra** corresponding to the invariant polynomial  $P$ , because, as described below in 6.5.1, connections with values in it pick out the Chern-form corresponding to  $P$ .

- For each degree  $2n-1$  transgression element  $\text{cs}$ , let  $\text{cs}_P(\mathfrak{g})$  be the  $L_\infty$ -algebra defined by

$$\text{CE}(\text{cs}_P(\mathfrak{g})) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \oplus \mathbb{R}[2n-2] \oplus \mathbb{R}[2n-1]), d_{\text{CE}(\text{ch}_P(\mathfrak{g}))}) \quad (231)$$

with

$$d_{\text{CE}(\text{cs}_P(\mathfrak{g}))}|_{\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])} = d_{W(\mathfrak{g})} \quad (232)$$

$$d_{\text{CE}(\text{cs}_P(\mathfrak{g}))}|_{\mathbb{R}[2n-2]} : b \mapsto -\text{cs} + c \quad (233)$$

$$d_{\text{CE}(\text{ch}_P(\mathfrak{g}))}|_{\mathbb{R}[2n-1]} : c \mapsto P, \quad (234)$$

where  $\{b\}$  and  $\{c\}$  denote the canonical bases of  $\mathbb{R}[2n-2]$  and  $\mathbb{R}[2n-1]$ , respectively. This we call the **Chern-Simons  $L_\infty$ -algebra** with respect to the transgression element  $\text{cs}$ , because, as described below in 6.5.1, connections with values in these come from (generalized) Chern-Simons forms.

The nilpotency of these differentials follows directly from the very definition of  $L_\infty$ -algebra cocycles and invariant polynomials.

**Proposition 20 (the string-like extensions)** *For each  $L_\infty$ -cocycle  $\mu \in \wedge^n(\mathfrak{g}^*)$  of degree  $n$ , the corresponding String-like extension sits in an exact sequence*

$$0 \longleftarrow \text{CE}(b^{n-1}u(1)) \longleftarrow \text{CE}(\mathfrak{g}_\mu) \longleftarrow \text{CE}(\mathfrak{g}) \longleftarrow 0 \quad (235)$$

Proof. The morphisms are the canonical inclusion and projection.  $\square$

**Proposition 21** *For  $\text{cs} \in W(\mathfrak{g})$  any transgression element interpolating between the cocycle  $\mu \in \text{CE}(\mathfrak{g})$  and the invariant polynomial  $P \in \wedge^\bullet(\mathfrak{g}[1]) \subset W(\mathfrak{g})$ , we obtain a homotopy-exact sequence*

$$\begin{array}{ccc} \text{CE}(\mathfrak{g}_\mu) & \longleftarrow & \text{CE}(\text{cs}_P(\mathfrak{g})) \longleftarrow \text{CE}(\text{ch}_P(\mathfrak{g})) \\ & \downarrow \simeq & \\ & W(\mathfrak{g}_\mu) & \end{array} \quad (236)$$

Here the isomorphism

$$f : W(\mathfrak{g}_\mu) \xrightarrow{\simeq} \text{CE}(\text{cs}_P(\mathfrak{g})) \quad (237)$$

is the identity on  $\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \oplus \mathbb{R}[n]$

$$f|_{\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \oplus \mathbb{R}[n]} = \text{Id} \quad (238)$$

and acts as

$$f|_{\mathbb{R}[n+1]} : b \mapsto c + \mu - \text{cs} \quad (239)$$

for  $b$  the canonical basis of  $\mathbb{R}[n]$  and  $c$  that of  $\mathbb{R}[n+1]$ . We check that this does respect the differentials

$$\begin{array}{ccc} b \xrightarrow{d_{W(\mathfrak{g}_\mu)}} -\mu + c & & c \xrightarrow{d_{W(\mathfrak{g}_\mu)}} \sigma\mu \\ \downarrow f & & \downarrow f \\ b \xrightarrow{d_{\text{CE}(\text{cs}_P(\mathfrak{g}))}} -\text{cs} + c & & c + \mu - \text{cs} \xrightarrow{d_{\text{CE}(\text{cs}_P(\mathfrak{g}))}} \sigma\mu \end{array} \quad (240)$$

Recall from definition 39 that  $\sigma$  is the canonical isomorphism  $\sigma : \mathfrak{g}^* \rightarrow \mathfrak{g}^*[1]$  extended by 0 to  $\mathfrak{g}^*[1]$  and then as a derivation to all of  $\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])$ . In the above the morphism between the Weil algebra of  $\mathfrak{g}_\mu$  and the Chevalley-Eilenberg algebra of  $\text{cs}_P(\mathfrak{g})$  is indeed an isomorphism (not just an equivalence). This isomorphism exhibits one of the main points to be made here: it makes manifest that the invariant polynomial  $P$  that is related by transgression to the cocycle  $\mu$  which induces  $\mathfrak{g}_\mu$  becomes exact with respect to  $\mathfrak{g}_\mu$ . This is the statement of proposition 23 below.



**$L_\infty$ -algebra cohomology and invariant polynomials of String-like extensions.** The  $L_\infty$ -algebra  $\mathfrak{g}_\mu$  obtained from an  $L_\infty$ -algebra  $\mathfrak{g}$  with an  $L_\infty$ -algebra cocycle  $\mu \in H^\bullet(\text{CE}(\mathfrak{g}))$  can be thought of as being obtained from  $\mathfrak{g}$  by “killing” a cocycle  $\mu$ . This is familiar from Sullivan models in rational homotopy theory.

**Proposition 22** *Let  $\mathfrak{g}$  be an ordinary semisimple Lie algebra and  $\mu$  a cocycle on it. Then*

$$H^\bullet(\text{CE}(\mathfrak{g}_\mu)) = H^\bullet(\text{CE}(\mathfrak{g}))/\langle \mu \rangle. \quad (241)$$

Accordingly, one finds that in cohomology the invariant polynomials on  $\mathfrak{g}_\mu$  are those of  $\mathfrak{g}$  except that the polynomial in transgression with  $\mu$  now suspends to 0.

**Proposition 23** *Let  $\mathfrak{g}$  be an  $L_\infty$ -algebra and  $\mu \in \text{CE}(\mathfrak{g})$  in transgression with the invariant polynomial  $P \in \text{inv}(\mathfrak{g})$ . Then with respect to the String-like extension  $\mathfrak{g}_\mu$  the polynomial  $P$  suspends to 0.*

Proof. Since  $\mu$  is a coboundary in  $\text{CE}(\mathfrak{g}_\mu)$ , this is a corollary of proposition 16.  $\square$

**Remark.** We will see in 7 that those invariant polynomials which suspend to 0 do actually not contribute to the characteristic classes. As we will also see there, this can be understood in terms of the invariant polynomials not with respect to the projection  $\text{CE}(\mathfrak{g}) \longleftarrow \text{W}(\mathfrak{g})$  but with respect to the projection  $\text{CE}(\mathfrak{g}) \longleftarrow \text{CE}(\text{cs}_P(\mathfrak{g}_\mu))$ , recalling from 21 that  $\text{W}(\mathfrak{g})$  is isomorphic to  $\text{cs}_P(\mathfrak{g})$ .

**Proposition 24** *For  $\mathfrak{g}$  any  $L_\infty$ -algebra with cocycle  $\mu$  of degree  $2n + 1$  in transgression with the invariant polynomial  $P$ , denote by  $\text{cs}_P(\mathfrak{g})_{\text{basic}}$  the DGCA of basic forms with respect to the canonical projection*

$$\text{CE}(\mathfrak{g}) \longleftarrow \text{CE}(\text{cs}_P(\mathfrak{g}_\mu)), \quad (242)$$

according to the general definition 9.

Then the cohomology of  $\text{cs}_P(\mathfrak{g})_{\text{basic}}$  is that of  $\text{inv}(\mathfrak{g})$  modulo  $P$ :

$$H^\bullet(\text{cs}_P(\mathfrak{g})_{\text{basic}}) \simeq H^\bullet(\text{inv}(\mathfrak{g}))/\langle P \rangle. \quad (243)$$

Proof. One finds that the vertical derivations on  $\text{CE}(\text{cs}_P(\mathfrak{g})) = \wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \oplus \mathbb{R}[n] \oplus \mathbb{R}[n + 1])$  are those that vanish on everything except the unshifted copy of  $\mathfrak{g}^*$ . Therefore the basic forms are those in  $\wedge^\bullet(\mathfrak{g}^*[1] \oplus \mathbb{R}[n] \oplus \mathbb{R}[n + 1])$  such that also their  $d_{\text{cs}_P(\mathfrak{g})}$ -differential is in that space. Hence all invariant  $\mathfrak{g}$ -polynomials are among them. But one of them now becomes exact, namely  $P$ .  $\square$

**Remark.** The first example below, definition 24, introduces the String Lie 2-algebra of an ordinary semisimple Lie algebra  $\mathfrak{g}$ , which gave all our String-like extensions its name. It is known, corollary 2 in [8] that the real cohomology of the classifying space of the 2-group integrating it is that of  $G = \exp(\mathfrak{g})$ , modulo the ideal generated by the class corresponding to  $P$ . Hence  $\text{CE}(\text{cs}_P(\mathfrak{g}))$  is an algebraic model for this space.

#### 6.4.1 Examples

**Ordinary central extensions.** Ordinary central extensions coming from a 2-cocycle  $\mu \in H^2(\text{CE}(\mathfrak{g}))$  of an ordinary Lie algebra  $\mathfrak{g}$  are a special case of the “string-like” extensions we are considering:

By definition 23 the  $L_\infty$ -algebra  $\mathfrak{g}_\mu$  is the Lie 1-algebra whose Chevalley-Eilenberg algebra is

$$\text{CE}(\mathfrak{g}_\mu) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathbb{R}[1]), d_{\text{CE}(\mathfrak{g}_\mu)}) \quad (244)$$

$$\begin{array}{ccc}
\mathrm{CE}(\mathfrak{g}_\mu) & \xrightarrow{=} & \mathrm{CE}(\mathfrak{g}_\mu) \\
\uparrow & & \uparrow \\
\mathrm{W}(\mathfrak{g}_\mu) & \xrightarrow{\simeq} & \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) \\
\uparrow & & \uparrow \\
\mathrm{inv}(\mathfrak{g}) = \mathrm{inv}(\mathfrak{g}_\mu) & \xrightarrow{\hookrightarrow} & \mathrm{cs}_P(\mathfrak{g})_{\mathrm{basic}}
\end{array}$$

$$H^\bullet(\mathrm{cs}_P(\mathfrak{g})_{\mathrm{basic}}) \simeq H^\bullet(\mathrm{inv}(\mathfrak{g}))/\langle P \rangle$$

Figure 8: The DGCA sequence playing the role of differential forms on the **universal (higher) String  $n$ -bundle** for a String-like extension  $\mathfrak{g}_\mu$ , definition 23, of an  $L_\infty$ -algebra  $\mathfrak{g}$  by a cocycle  $\mu$  of odd degree in transgression with an invariant polynomial  $P$ . Compare with figure 1. In  $H^\bullet(\mathrm{inv}(\mathfrak{g}_\mu)) = H^\bullet(\mathrm{W}(\mathfrak{g}_\mu)_{\mathrm{basic}})$  the class of  $P$  is still contained, but suspends to 0, according to proposition 23. In  $H^\bullet(\mathrm{cs}_P(\mathfrak{g})_{\mathrm{basic}})$  the class of  $P$  vanishes, according to proposition 24. The isomorphism  $\mathrm{W}(\mathfrak{g}_\mu) \simeq \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g}))$  is from proposition 21. For  $\mathfrak{g}$  an ordinary semisimple Lie algebra and  $\mathfrak{g}_\mu$  the ordinary String extension coming from the canonical 3-cocycle, this corresponds to the fact that the classifying space of the String 2-group [7, 43] has the cohomology of the classifying space of the underlying group, modulo the first Pontrajagin class [8].

where

$$d_{\mathrm{CE}(\mathfrak{g}_\mu)}|_{\mathfrak{g}^*} = d_{\mathrm{CE}(\mathfrak{g})} \quad (245)$$

and

$$d_{\mathrm{CE}(\mathfrak{g}_\mu)}|_{\mathbb{R}[1]} : b \mapsto \mu \quad (246)$$

for  $b$  the canonical basis of  $\mathbb{R}[1]$ . (Recall that in our conventions  $\mathfrak{g}$  is in degree 1).

This is indeed the Chevalley-Eilenberg algebra corresponding to the Lie bracket

$$[(x, c), (x', c')] = ([x, x'], \mu(x, x')) \quad (247)$$

(for all  $x, x' \in \mathfrak{g}$ ,  $c, c' \in \mathbb{R}$ ) on the centrally extended Lie algebra.

### The String Lie 2-algebra.

**Definition 24** Let  $\mathfrak{g}$  be a semisimple Lie algebra and  $\mu = \langle \cdot, [\cdot, \cdot] \rangle$  the canonical 3-cocycle on it. Then

$$\mathrm{string}(\mathfrak{g}) \quad (248)$$

is defined to be the strict Lie 2-algebra coming from the crossed module

$$(\hat{\Omega}\mathfrak{g} \rightarrow P\mathfrak{g}), \quad (249)$$

where  $P\mathfrak{g}$  is the Lie algebra of based paths in  $\mathfrak{g}$  and  $\hat{\Omega}\mathfrak{g}$  the Lie algebra of based loops in  $\mathfrak{g}$ , with central extension induced by  $\mu$ . Details are in [7].

**Proposition 25 ([7])** The Lie 2-algebra  $\mathfrak{g}_\mu$  obtained from  $\mathfrak{g}$  and  $\mu$  as in definition 23 is equivalent to the strict string Lie 2-algebra

$$\mathfrak{g}_\mu \simeq \mathrm{string}(\mathfrak{g}). \quad (250)$$

This means there are morphisms  $\mathfrak{g}_\mu \rightarrow \text{string}(\mathfrak{g})$  and  $\text{string}(\mathfrak{g}) \rightarrow \mathfrak{g}_\mu$  whose composite is the identity only up to homotopy

$$\begin{array}{ccc} \mathfrak{g}_\mu & \xrightarrow{\quad} & \text{string}(\mathfrak{g}) \xrightarrow{\quad} \mathfrak{g}_\mu \\ & \Downarrow = & \\ & \text{Id} & \end{array} \quad \begin{array}{ccc} \text{string}(\mathfrak{g}) & \xrightarrow{\quad} & \mathfrak{g}_\mu \xrightarrow{\quad} \text{string}(\mathfrak{g}) \\ & \Downarrow \eta & \\ & \text{Id} & \end{array} \quad (251)$$

We call  $\mathfrak{g}_\mu$  the *skeletal* and  $\text{string}(\mathfrak{g})$  the *strict* version of the String Lie 2-algebra.

### The Fivebrane Lie 6-algebra

**Definition 25** Let  $\mathfrak{g} = \mathfrak{so}(n)$  and  $\mu$  the canonical 7-cocycle on it. Then

$$\text{fivebrane}(\mathfrak{g}) \quad (252)$$

is defined to be the strict Lie 7-algebra which is equivalent to  $\mathfrak{g}_\mu$

$$\mathfrak{g}_\mu \simeq \text{fivebrane}(\mathfrak{g}). \quad (253)$$

A Lie  $n$ -algebra is *strict* if it corresponds to a differential graded Lie algebra on a vector space in degree 1 to  $n$ . (Recall our grading conventions from 6.1.)

**Remark.** It is a major open problem to identify the strict  $\text{fivebrane}(\mathfrak{g})$ . Proposition 25 suggests that it might involve hyperbolic Kac-Moody algebras and/or the torus algebra of  $\mathfrak{g}$ , since these would seem to be what comes beyond the affine Kac-Moody algebras relevant for  $\text{string}(n)$ .

### The BF-theory Lie 3-algebra.

**Definition 26** For  $\mathfrak{g}$  any ordinary Lie algebra with bilinear invariant symmetric form  $\langle \cdot, \cdot \rangle \in \text{inv}(\mathfrak{g})$  in transgression with the 3-cocycle  $\mu$ , and for  $\mathfrak{h} \xrightarrow{t} \mathfrak{g}$  a strict Lie 2-algebra based on  $\mathfrak{g}$ , denote by

$$\hat{\mu} := d_{\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} \mu \quad (254)$$

the corresponding exact 4-cocycle on  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  discussed in 6.3.1. Then we call the string-like extended Lie 3-algebra

$$\mathfrak{bf}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) := (\mathfrak{h} \xrightarrow{t} \mathfrak{g})_{\hat{\mu}} \quad (255)$$

the corresponding BF-theory Lie 3-algebra.

The terminology here will become clear once we describe in 8.3.1 and 9.1.1 how the BF-theory action functional discussed in 6.6.1 arises as the parallel 4-transport given by the  $b^3\mathfrak{u}(1)$ -4-bundle which arises as the obstruction to lifting  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ -2-descent objects to  $\mathfrak{bf}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ -3-descent objects.

## 6.5 $L_\infty$ -algebra valued forms

Consider an ordinary Lie algebra  $\mathfrak{g}$  valued connection form  $A$  regarded as a linear map  $\mathfrak{g}^* \rightarrow \Omega^1(Y)$ . Since  $\text{CE}(\mathfrak{g})$  is free as a graded commutative algebra, this linear map extends uniquely to a morphism of graded commutative algebras, though not in general of differential graded commutative algebra. In fact, the deviation is measured by the *curvature*  $F_A$  of the connection. However, the differential in  $W(\mathfrak{g})$  is precisely such that the connection does extend to a morphism of differential graded-commutative algebras

$$W(\mathfrak{g}) \rightarrow \Omega^\bullet(Y). \quad (256)$$

This implies that a good notion of a  $\mathfrak{g}$ -valued differential form on a smooth space  $Y$ , for  $\mathfrak{g}$  any  $L_\infty$ -algebra, is a morphism of differential graded-commutative algebras from the Weil algebra of  $\mathfrak{g}$  to the algebra of differential forms on  $Y$ .

**Definition 27 (g-valued forms)** For  $Y$  a smooth space and  $\mathfrak{g}$  an  $L_\infty$ -algebra, we call

$$\Omega^\bullet(Y, \mathfrak{g}) := \text{Hom}_{\text{dgc-Alg}}(W(\mathfrak{g}), \Omega^\bullet(Y)) \quad (257)$$

the space of  $\mathfrak{g}$ -valued differential forms on  $X$ .

**Definition 28 (curvature)** We write  $\mathfrak{g}$ -valued differential forms as

$$(\Omega^\bullet(Y) \xleftarrow{(A, F_A)} W(\mathfrak{g})) \in \Omega^\bullet(Y, \mathfrak{g}), \quad (258)$$

where  $F_A$  denotes the restriction to the shifted copy  $\mathfrak{g}^*[1]$  given by

$$\text{curv} : (\Omega^\bullet(Y) \xleftarrow{(A, F_A)} W(\mathfrak{g})) \mapsto (\Omega^\bullet(Y) \xleftarrow{(A, F_A)} W(\mathfrak{g}) \xleftarrow{F_A} \mathfrak{g}^*[1]). \quad (259)$$

$F_A$  we call the curvature of  $A$ .

**Proposition 26** The  $\mathfrak{g}$ -valued differential form  $\Omega^\bullet(Y) \xleftarrow{(A, F_A)} W(\mathfrak{g})$  factors through  $\text{CE}(\mathfrak{g})$  precisely when its curvature  $F_A$  vanishes.

$$\begin{array}{ccc} \text{CE}(\mathfrak{g}) & \xleftarrow{\quad} & W(\mathfrak{g}) \\ \text{dotted} \downarrow (A, F_A=0) & & \downarrow (A, F_A) \\ \Omega^\bullet(Y) & \xlongequal{\quad} & \Omega^\bullet(Y) \end{array} \quad (260)$$

In this case we say that  $A$  is **flat**. Hence the space of flat  $\mathfrak{g}$ -valued forms is

$$\Omega_{\text{flat}}^\bullet(Y, \mathfrak{g}) \simeq \text{Hom}_{\text{dgc-Alg}}(\text{CE}(\mathfrak{g}), \Omega^\bullet(Y)). \quad (261)$$

**Bianchi identity.** Recall from 6.1 that the Weil algebra  $W(\mathfrak{g})$  of an  $L_\infty$ -algebra  $\mathfrak{g}$  is the same as the Chevalley-Eilenberg algebra  $\text{CE}(\text{inn}(\mathfrak{g}))$  of the  $L_\infty$ -algebra of inner derivation of  $\mathfrak{g}$ . It follows that  $\mathfrak{g}$ -valued differential forms on  $Y$  are the same as *flat*  $\text{inn}(\mathfrak{g})$ -valued differential forms on  $Y$ :

$$\Omega^\bullet(Y, \mathfrak{g}) = \Omega_{\text{flat}}^\bullet(\text{inn}(\mathfrak{g})). \quad (262)$$

By the above definition of curvature, this says that the curvature  $F_A$  of a  $\mathfrak{g}$ -valued connection  $(A, F_A)$  is itself a flat  $\text{inn}(\mathfrak{g})$ -valued connection. This is the generalization of the ordinary *Bianchi identity* to  $L_\infty$ -algebra valued forms.

**Definition 29** Two  $\mathfrak{g}$ -valued forms  $A, A' \in \Omega^\bullet(Y, \mathfrak{g})$  are called **(gauge) equivalent** precisely if they are related by a vertical concordance, i.e. by a concordance, such that the corresponding derivation  $\rho$  from proposition 10 is vertical, in the sense of definition 8.

### 6.5.1 Examples

**1. Ordinary Lie-algebra valued 1-forms.** We have already mentioned ordinary Lie algebra valued 1-forms in this general context in 2.1.3.

**2. Forms with values in shifted  $b^{n-1}\mathfrak{u}(1)$**

A  $b^{n-1}\mathfrak{u}(1)$ -valued form is nothing but an ordinary  $n$ -form  $A \in \Omega^n(Y)$ :

$$\Omega^\bullet(b^{n-1}\mathfrak{u}(1), Y) \simeq \Omega^n(Y). \quad (263)$$

A flat  $b^{n-1}\mathfrak{u}(1)$ -valued form is precisely a closed  $n$ -form.

$$\begin{array}{ccc} \mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \longleftarrow & \mathrm{W}(b^{n-1}\mathfrak{u}(1)) \\ \downarrow \scriptstyle (A) & & \downarrow \scriptstyle (A, F_A) \\ dA=0 & & A=dA \\ \downarrow & \xrightarrow{=} & \downarrow \\ \Omega^\bullet(Y) & & \Omega^\bullet(Y) \end{array}$$

**3. Crossed module valued forms.** Let  $\mathfrak{g}_{(2)} = (\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  be a strict Lie 2-algebra coming from a crossed module. Then a  $\mathfrak{g}_{(2)}$ -valued form is an ordinary  $\mathfrak{g}$ -valued 1-form  $A$  and an ordinary  $\mathfrak{h}$ -valued 2-form  $B$ . The corresponding curvature is an ordinary  $\mathfrak{g}$ -valued 2-form  $\beta = F_A + t(B)$  and an ordinary  $\mathfrak{h}$ -valued 3-form  $H = d_A B$ . This is denoted by the right vertical arrow in the following diagram.

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) & \longleftarrow & \mathrm{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \\ \downarrow \scriptstyle (A, B) & & \downarrow \scriptstyle (A, B, \beta, H) \\ F_A + t(B) = 0 & & \begin{array}{l} \beta = F_A + t(B) \\ H = d_A B \end{array} \\ \downarrow & \xrightarrow{=} & \downarrow \\ \Omega^\bullet(Y) & & \Omega^\bullet(Y) \end{array} \quad (264)$$

Precisely if the curvature components  $\beta$  and  $H$  vanish, does this morphism on the right factor through  $\mathrm{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$ , which is indicated by the left vertical arrow of the above diagram.

**4. String Lie  $n$ -algebra valued forms.** For  $\mathfrak{g}$  an ordinary Lie algebra and  $\mu$  a degree  $(2n+1)$ -cocycle on  $\mathfrak{g}$  the situation is captured by the following diagram

$$\begin{array}{ccccccc} & \text{String-like} & & \text{Chern-Simons} & & \text{Chern} & \\ & & & & & & \\ & 1 & & 2n & & 2n+1 & & 2n+1 \\ & & & & & & & \\ \mathrm{CE}(\mathfrak{g}) & \hookrightarrow & \mathrm{CE}(\mathfrak{g}_\mu) & \longleftarrow & \mathrm{CE}(\mathrm{cs}_P(\mathfrak{g})) & \longleftarrow & \mathrm{CE}(\mathrm{ch}_P(\mathfrak{g})) \\ \downarrow \scriptstyle (A) & & \downarrow \scriptstyle (A, B) & & \downarrow \scriptstyle (A, B, C) & & \downarrow \scriptstyle (A, C) \\ F_A=0 & & \begin{array}{l} F_A=0 \\ dB + \mathrm{CS}_k(A)=0 \end{array} & & C = dB + \mathrm{CS}_P(A) & & dC = k((F_A)^{n+1}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega^\bullet(Y) & \xrightarrow{=} & \Omega^\bullet(Y) & \xrightarrow{=} & \Omega^\bullet(Y) & \xrightarrow{=} & \Omega^\bullet(Y) \end{array} \quad (265)$$

Here  $\mathrm{CS}_P(A)$  denotes the Chern-Simons form such that  $d\mathrm{CS}_P(A) = P(F_A)$ , given by the specific contracting homotopy.

The standard example is that corresponding to the ordinary String-extension.

$$\begin{array}{ccccccc}
\mathrm{CE}(\mathfrak{g}) & \hookrightarrow & \mathrm{CE}(\mathrm{string}(\mathfrak{g})) & \longleftarrow & \mathrm{W}(\mathrm{string}_k(\mathfrak{g})) & & \\
\parallel & & \parallel \simeq & & \parallel \simeq & & \\
\mathrm{CE}(\mathfrak{g}) & \hookrightarrow & \mathrm{CE}(\mathfrak{g}_\mu) & \longleftarrow & \mathrm{CE}(\mathrm{cs}_k(\mathfrak{g})) & \longleftarrow & \mathrm{CE}(\mathrm{ch}_P(\mathfrak{g})) \\
(A) \downarrow & & (A, B) \downarrow & & (A, B, C) \downarrow & & (A, C) \downarrow \\
F_A=0 & & \begin{array}{c} F_A=0 \\ dB + \mathrm{CS}_P(A)=0 \end{array} & & C = dB + \mathrm{CS}_P(A) & & dC = \langle F_A \wedge F_A \rangle \\
\Omega^\bullet(Y) & \xlongequal{\quad} & \Omega^\bullet(Y) & \xlongequal{\quad} & \Omega^\bullet(Y) & \xlongequal{\quad} & \Omega^\bullet(Y)
\end{array}
\tag{266}$$

In the above,  $\mathfrak{g}$  is semisimple with invariant bilinear form  $P = \langle \cdot, \cdot \rangle$  related by transgression to the 3-cocycle  $\mu = \langle \cdot, [\cdot, \cdot] \rangle$ . Then the Chern-Simons 3-form for any  $\mathfrak{g}$ -valued 1-form  $A$  is

$$\mathrm{CS}_{\langle \cdot, \cdot \rangle}(A) = \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle.
\tag{267}$$

### 5. Fields of 11-dimensional supergravity.

While we shall not discuss it in detail here, it is clear that the entire discussion we give has a straightforward generalization to *super*  $L_\infty$ -algebras, obtained simply by working entirely within the category of super vector spaces (the category of  $\mathbb{Z}_2$ -graded vector spaces equipped with the unique non-trivial symmetric braiding on it, which introduces a sign whenever two odd-graded vector spaces are interchanged).

A glance at the definitions shows that, up to mere differences in terminology, the theory of “FDA”s (“free differential algebras”) considered in [4, 24] is nothing but that of what we call qDGCAs here: quasi-free differential graded commutative algebras.

Using that and the interpretation of qDGCAs in terms of  $L_\infty$ -algebras, one can translate everything said in [4, 24] into our language here to obtain the following statement:

The field content of 11-dimensional supergravity is nothing but a  $\mathfrak{g}$ -valued form, for

$$\mathfrak{g} = \mathfrak{sugra}(10, 1)
\tag{268}$$

the Lie 3-algebra which is the string-like extension

$$0 \rightarrow b^2\mathfrak{u}(1) \rightarrow \mathfrak{sugra}(10, 1) \rightarrow \mathfrak{siso}(10, 1) \rightarrow 0
\tag{269}$$

of the *super-Poincaré* Lie algebra in 10+1 dimensions, coming from a certain 4-cocycle on that.

$$\begin{array}{ccc}
\mathrm{CE}(\mathfrak{sugra}(10, 1)) & \longleftarrow & \mathrm{W}(\mathfrak{sugra}(10, 1)) \\
\downarrow \text{(flat connection, parallel gravitino, vanishing supertorsion)} & & \downarrow \text{(graviton } g, \text{ gravitino } \Psi, \text{ 3-form } C_3) \\
\Omega^\bullet(Y) & \xlongequal{\quad} & \Omega^\bullet(Y)
\end{array}
\tag{270}$$

While we shall not further pursue this here, this implies the following two interesting issues.

- It is known in string theory [28] that the supergravity 3-form in fact consists of three parts: two Chern-Simons parts for an  $\mathfrak{e}_8$  and for a  $\mathfrak{so}(10, 1)$ -connection, as well as a further fermionic part, coming precisely from the 4-cocycle that governs  $\mathfrak{sugra}(10, 1)$ . As we discuss in 8 and 9, the two Chern-Simons components can be understood in terms of certain Lie 3-algebra connections coming from the Chern-Simons Lie 3-algebra  $\mathrm{cs}_P(\mathfrak{g})$  from definition 23. It hence seems that there should be a Lie  $n$ -algebra which nicely unifies  $\mathrm{cs}_P(\mathfrak{e}_8)$ ,  $\mathrm{cs}_P(\mathfrak{so}(10, 1)_8)$  and  $\mathfrak{sugra}(10, 1)$ . This remains to be discussed.

- The discussion in 7 shows how to obtain from  $\mathfrak{g}$ -valued forms globally defined connections on possibly nontrivial  $\mathfrak{g}$ - $n$ -bundles. Applied to  $\mathfrak{sugra}(10, 1)$  this should yield a *global* description of the supergravity field content, which extends the local field content considered in [4, 24] in the way a connection in a possibly nontrivial Yang-Mills bundle generalizes a Lie algebra valued 1-form. This should for instance allow to discuss supergravity *instanton* solutions.

## 6.6 $L_\infty$ -algebra characteristic forms

**Definition 30** For  $\mathfrak{g}$  any  $L_\infty$  algebra and

$$\Omega^\bullet(Y) \xleftarrow{(A, F_A)} W(\mathfrak{g}) \quad (271)$$

any  $\mathfrak{g}$ -valued differential form, we call the composite

$$\begin{array}{ccc} & \{P(F_A)\} & \\ & \curvearrowright & \\ \Omega^\bullet(Y) & \xleftarrow{(A, F_A)} W(\mathfrak{g}) & \xleftarrow{\quad} \text{inv}(\mathfrak{g}) \end{array} \quad (272)$$

the collection of **invariant forms** of the  $\mathfrak{g}$ -valued form  $A$ . We call the deRham classes  $[P(F_A)]$  of the characteristic forms arising as the image of closed invariant polynomials

$$\begin{array}{ccc} & \{P_i(F_A)\} & \\ & \curvearrowright & \\ \Omega^\bullet(Y) & \xleftarrow{(A, F_A)} W(\mathfrak{g}) & \xleftarrow{\quad} \text{inv}(\mathfrak{g}) \end{array} \quad (273)$$

$$H_{\text{dR}}^\bullet(Y) \xleftarrow{\{[P(F_A)]\}} H^\bullet(\text{inv}(\mathfrak{g}))$$

the collection of **characteristic classes** of the  $\mathfrak{g}$ -valued form  $A$ .

Recall from 6.3 that for ordinary Lie algebras all invariant polynomials are closed, while for general  $L_\infty$ -algebras it is only true that their  $d_{W(\mathfrak{g})}$ -differential is horizontal. Notice that  $Y$  will play the role of a cover of some space  $X$  soon, and that characteristic forms really live down on  $X$ . We will see shortly a constraint imposed which makes the characteristic forms descend down from the  $Y$  here to such an  $X$ .

**Proposition 27** Under gauge transformations as in definition 29, characteristic classes are invariant.

Proof. This follows from proposition 10: By that proposition, the derivative of the concordance form  $\hat{A}$  along the interval  $I = [0, 1]$  is a chain homotopy

$$\frac{d}{dt} \hat{A}(P) = [d, \iota_X]P = d\tau(P) + \iota_X(d_{W(\mathfrak{g})}P). \quad (274)$$

By definition of gauge-transformations,  $\iota_X$  is vertical. By definition of basic forms,  $P$  is both in the kernel of  $\iota_X$  as well as in the kernel of  $\iota_X \circ d$ . Hence the right hand vanishes.  $\square$

### 6.6.1 Examples

**Characteristic forms of  $b^{n-1}\mathfrak{u}(1)$ -valued forms.**

**Proposition 28** A  $b^{n-1}\mathfrak{u}(1)$ -valued form  $\Omega^\bullet(Y) \xleftarrow{A} W(b^{n-1}\mathfrak{u}(1))$  is precisely an  $n$ -form on  $Y$ :

$$\Omega^\bullet(Y, b^{n-1}\mathfrak{u}(1)) \simeq \Omega^n(Y). \quad (275)$$

If two such  $b^{n-1}\mathfrak{u}(1)$ -valued forms are gauge equivalent according to definition 29, then their curvatures coincide

$$(\Omega^\bullet(Y) \xleftarrow{A} W(b^{n-1}\mathfrak{u}(1))) \sim (\Omega^\bullet(Y) \xleftarrow{A'} W(b^{n-1}\mathfrak{u}(1))) \Rightarrow dA = dA'. \quad (276)$$

**BF-theory.** We demonstrate that the expression known in the literature as the *action functional for BF-theory with cosmological term* is the integral of an invariant polynomial for  $\mathfrak{g}$ -valued differential forms where  $\mathfrak{g}$  is a Lie 2-algebra. Namely, let  $\mathfrak{g}_{(2)} = (\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  be any strict Lie 2-algebra as in 6.1. Let

$$P = \langle \cdot, \cdot \rangle \quad (277)$$

be an invariant bilinear form on  $\mathfrak{g}$ , hence a degree 2 invariant polynomial on  $\mathfrak{g}$ . According to proposition 18,  $P$  therefore also is an invariant polynomial on  $\mathfrak{g}_{(2)}$ .

Now for  $(A, B)$  a  $\mathfrak{g}_{(2)}$ -valued differential form on  $X$ , as in the example in 6.5,

$$\Omega^\bullet(Y) \xleftarrow{((A,B),(\beta,H))} W(\mathfrak{g}_{(2)}) , \quad (278)$$

one finds

$$\begin{array}{ccc} \Omega^\bullet(Y) & \xleftarrow{((A,B),(\beta,H))} W(\mathfrak{g}_{(2)}) & \xleftarrow{\text{inv}(\mathfrak{g}_{(2)})} \\ & \searrow \text{Pontryagin term} & \nearrow \text{BF-term} \\ & P \mapsto \langle \beta, \beta \rangle & \end{array} \quad (279)$$

so that the corresponding characteristic form is the 4-form

$$P(\beta, H) = \langle \beta \wedge \beta \rangle = \langle (F_A + t(B)) \wedge (F_A + t(B)) \rangle. \quad (280)$$

Collecting terms as

$$P(\beta, H) = \underbrace{\langle F_A \wedge F_A \rangle}_{\text{Pontryagin term}} + 2 \underbrace{\langle t(B) \wedge F_A \rangle}_{\text{BF-term}} + \underbrace{\langle t(B) \wedge t(B) \rangle}_{\text{"cosmological constant"}} \quad (281)$$

we recognize the Lagrangian for topological Yang-Mills theory and BF theory with cosmological term.

For  $X$  a compact 4-manifold, the corresponding action functional

$$S : \Omega^\bullet(X, \mathfrak{g}_{(2)}) \rightarrow \mathbb{R} \quad (282)$$

sends  $\mathfrak{g}_{(2)}$ -valued 2-forms to the integral of this 4-form

$$(A, B) \mapsto \int_X (\langle F_A \wedge F_A \rangle + 2 \langle t(B) \wedge F_A \rangle + \langle t(B) \wedge t(B) \rangle). \quad (283)$$

The first term here is usually not considered an intrinsic part of BF-theory, but its presence does not affect the critical points of  $S$ .

The critical points of  $S$ , i.e. the  $\mathfrak{g}_{(2)}$ -valued differential forms on  $X$  that satisfy the equations of motion defined by the action  $S$ , are given by the equation

$$\beta := F_A + t(B) = 0. \quad (284)$$

Notice that this implies

$$d_A t(B) = 0 \quad (285)$$



but does not constrain the full 3-curvature

$$H = d_A B \quad (286)$$

to vanish. In other words, the critical points of  $S$  are precisely the *fake flat*  $\mathfrak{g}_{(2)}$ -valued forms which precisely integrate to strict parallel transport 2-functors [37, 69, 9].

While the 4-form  $\langle \beta \wedge \beta \rangle$  looks similar to the Pontrjagin 4-form  $\langle F_A \wedge F_A \rangle$  for an ordinary connection 1-form  $A$ , one striking difference is that  $\langle \beta \wedge \beta \rangle$  is, in general, not closed. Instead, according to equation 211, we have

$$d\langle \beta \wedge \beta \rangle = 2\langle \beta \wedge t(H) \rangle. \quad (287)$$

**Remark.** Under the equivalence [7] of the skeletal String Lie 2-algebra to its strict version, recalled in proposition 25, the characteristic forms for strict Lie 2-algebras apply also to one of our central objects of interest here, the String 2-connections. But a little care needs to be exercised here, because the strict version of the String Lie 2-algebra is no longer finite dimensional.

**Remark.** Our interpretation above of BF-theory as a gauge theory for Lie 2-algebras is not unrelated to, but different from the one considered in [37, 38]. There only the Lie 2-algebra coming from the infinitesimal crossed module  $(|\mathfrak{g}| \xrightarrow{0} \mathfrak{g} \xrightarrow{\text{ad}} \text{der}(\mathfrak{g}))$  (for  $\mathfrak{g}$  any ordinary Lie algebra and  $|\mathfrak{g}|$  its underlying vector space, regarded as an abelian Lie algebra) is considered, and the action is restricted to the term  $\int \langle F_A \wedge B \rangle$ . We can regard the above discussion as a generalization of this approach to arbitrary Lie 2-algebras. Standard BF-theory (with “cosmological” term) is reproduced with the above Lagrangian by using the Lie 2-algebra  $\text{inn}(\mathfrak{g})$  corresponding to the infinitesimal crossed module  $(\mathfrak{g} \xrightarrow{\text{Id}} \mathfrak{g} \xrightarrow{\text{ad}} \text{der}(\mathfrak{g}))$  discussed in 6.1.1.

## 7 $L_\infty$ -algebra Cartan-Ehresmann connections

We will now combine all of the above ingredients to produce a definition of  $\mathfrak{g}$ -valued connections. As we shall explain, the construction we give may be thought of as a generalization of the notion of a Cartan-Ehresmann connection, which is given by a Lie algebra-valued 1-form on the total space of a bundle over base space satisfying two conditions:

- first Cartan-Ehresmann condition: on the fibers the connection form restricts to a *flat* canonical form
- second Cartan-Ehresmann condition: under vertical flows the connections transforms nicely, in such a way that its characteristic forms descend down to base space.

We will essentially interpret these two conditions as a pullback of the universal  $\mathfrak{g}$ -bundle, in its DGC-algebraic incarnation as given in equation 178.

The definition we give can also be seen as the Lie algebraic image of a similar construction involving locally trivializable transport  $n$ -functors [9, 70], but this shall not be further discussed here.

### 7.1 $\mathfrak{g}$ -Bundle descent data

**Definition 31 ( $\mathfrak{g}$ -bundle descent data)** *Given a Lie  $n$ -algebra  $\mathfrak{g}$ , a  $\mathfrak{g}$ -bundle descent object on  $X$  is a pair  $(Y, A_{\text{vert}})$  consisting of a choice of surjective submersion  $\pi : Y \rightarrow X$  with connected fibers (this condition will be dropped when we extend to  $\mathfrak{g}$ -connection descent objects in 7.2) together with a morphism of dg-algebras*

$$\Omega_{\text{vert}}^\bullet(Y) \xleftarrow{A_{\text{vert}}} \text{CE}(\mathfrak{g}) . \quad (288)$$

*Two such descent objects are taken to be equivalent*

$$(\Omega_{\text{vert}}^\bullet(Y) \xleftarrow{A_{\text{vert}}} \text{CE}(\mathfrak{g})) \sim (\Omega_{\text{vert}}^\bullet(Y') \xleftarrow{A'_{\text{vert}}} \text{CE}(\mathfrak{g})) \quad (289)$$

precisely if their pullbacks  $\pi_1^* A_{\text{vert}}$  and  $\pi_2^* A'_{\text{vert}}$  to the common refinement

$$\begin{array}{ccc} Y \times_X Y' & \xrightarrow{\pi_1} & Y \\ \pi_2 \downarrow & & \downarrow \pi \\ Y' & \xrightarrow{\pi'} & X \end{array} \quad (290)$$

are concordant in the sense of definition 18.

Thus two such descent objects  $A_{\text{vert}}, A'_{\text{vert}}$  on the same  $Y$  are equivalent if there is  $\eta_{\text{vert}}^*$  such that

$$\begin{array}{ccc} & \xrightarrow{A_{\text{vert}}} & \\ \Omega_{\text{vert}}^\bullet(Y) & \xleftarrow[t^*]{s^*} \Omega_{\text{vert}}^\bullet(Y \times I) & \xleftarrow{\eta_{\text{vert}}^*} \text{CE}(\mathfrak{g}) \\ & \xleftarrow{A'_{\text{vert}}} & \end{array} \quad (291)$$

Recall from the discussion in 2.2.1 that the surjective submersions here play the role of open covers of  $X$ .

### 7.1.1 Examples

**Example: ordinary  $G$ -bundles.** The following example is meant to illustrate how the notion of descent data with respect to a Lie algebra  $\mathfrak{g}$  as defined here can be related to the ordinary notion of descent data with respect to a Lie group  $G$ . Consider the case where  $\mathfrak{g}$  is an ordinary Lie (1-)algebra. A  $\mathfrak{g}$ -cocycle then is a surjective submersion  $\pi : Y \rightarrow X$  together with a  $\mathfrak{g}$ -valued flat vertical 1-form  $A_{\text{vert}}$  on  $Y$ . Assume the fiber of  $\pi : Y \rightarrow X$  to be simply connected. Then for any two points  $(y, y') \in Y \times_X Y$  in the same fiber we obtain an element  $g(y, y') \in G$ , where  $G$  is the simply connected Lie group integrating  $\mathfrak{g}$ , by choosing any path  $y \xrightarrow{\gamma} y'$  in the fiber connecting  $y$  with  $y'$  and forming the parallel transport determined by  $A_{\text{vert}}$  along this path

$$g(y, y') := P \exp \left( \int_{\gamma} A_{\text{vert}} \right). \quad (292)$$

By the flatness of  $A_{\text{vert}}$  and the assumption that the fibers of  $Y$  are simply connected

- $g : Y \times_X Y \rightarrow G$  is well defined (does not depend on the choice of paths), and
- satisfies the cocycle condition for  $G$ -bundles

$$g : \begin{array}{ccc} & y' & \\ & \nearrow & \searrow \\ y & \xrightarrow{\quad} & y'' \end{array} \mapsto \begin{array}{ccc} & \bullet & \\ g(y, y') \nearrow & & \searrow g(y', y'') \\ \bullet & \xrightarrow{g(y, y'')} & \bullet \end{array} \quad (293)$$

Any such cocycle  $g$  defines a  $G$ -principal bundle. Conversely, every  $G$ -principal bundle  $P \rightarrow X$  gives rise to a structure like this by choosing  $Y := P$  and letting  $A_{\text{vert}}$  be the canonical invariant  $\mathfrak{g}$ -valued vertical 1-form on  $Y = P$ . Then suppose  $(Y, A_{\text{vert}})$  and  $(Y, A'_{\text{vert}})$  are two such cocycles defined on the same  $Y$ , and let  $(\hat{Y} := Y \times I, \hat{A}_{\text{vert}})$  be a concordance between them. Then, for every path

$$y \times \{0\} \xrightarrow{\gamma} y \times \{1\} \quad (294)$$

connecting the two copies of a point  $y \in Y$  over the endpoints of the interval, we again obtain a group element

$$h(y) := P \exp \left( \int_{\gamma} \hat{A}_{\text{vert}} \right). \quad (295)$$

By the flatness of  $\hat{A}$ , this is

- well defined in that it is independent of the choice of path;
- has the property that for all  $(y, y') \in Y \times_X Y$  we have

$$\begin{array}{ccc}
y \times \{0\} & \xrightarrow{\quad} & y \times \{1\} \\
\downarrow & & \downarrow \\
y' \times \{0\} & \xrightarrow{\quad} & y' \times \{1\}
\end{array}
\quad \mapsto \quad
\begin{array}{ccc}
\bullet & \xrightarrow{h(y)} & \bullet \\
\downarrow g(y, y') & & \downarrow g'(y, y') \\
\bullet & \xrightarrow{h(y')} & \bullet
\end{array} . \quad (296)$$

Therefore  $h$  is a gauge transformation between  $g$  and  $g'$ , as it should be.

Note that there is no holonomy since the fibers are assumed to be simply connected in this example.

**Abelian gerbes, Deligne cohomology and  $(b^{n-1}\mathfrak{u}(1))$ -descent objects** For the case that the  $L_\infty$ -algebra in question is shifted  $\mathfrak{u}(1)$ , i.e.  $\mathfrak{g} = b^{n-1}\mathfrak{u}(1)$ , classes of  $\mathfrak{g}$ -descent objects on  $X$  should coincide with classes of “line  $n$ -bundles”, i.e. with classes of abelian  $(n-1)$ -gerbes on  $X$ , hence with elements in  $H^n(X, \mathbb{Z})$ . In order to understand this, we relate classes of  $b^{n-1}\mathfrak{u}(1)$ -descent objects to Deligne cohomology. We recall Deligne cohomology for a fixed surjective submersion  $\pi : Y \rightarrow X$ . For comparison with some parts of the literature, the reader should choose  $Y$  to be the disjoint union of sets of a good cover of  $X$ . More discussion of this point is in 5.3.1.

The following definition should be thought of this way: a collection of  $p$ -forms on fiberwise intersections of a surjective submersion  $Y \rightarrow X$  are given. The 0-form part defines an  $n$ -bundle (an  $(n-1)$ -gerbe) itself, while the higher forms encode a connection on that  $n$ -bundle.

**Definition 32 (Deligne cohomology)** *Deligne cohomology can be understood as the cohomology on differential forms on the simplicial space  $Y^\bullet$  given by a surjective submersion  $\pi : Y \rightarrow X$ , where the complex of forms is taken to start as*

$$0 \longrightarrow C^\infty(Y^{[n]}, \mathbb{R}/\mathbb{Z}) \xrightarrow{d} \Omega^1(Y^{[n]}, \mathbb{R}) \xrightarrow{d} \Omega^2(Y^{[n]}, \mathbb{R}) \xrightarrow{d} \cdots , \quad (297)$$

where the first differential, often denoted  $d\log$  in the literature, is evaluated by acting with the ordinary differential on any  $\mathbb{R}$ -valued representative of a  $U(1) \simeq \mathbb{R}/\mathbb{Z}$ -valued function.

More in detail, given a surjective submersion  $\pi : Y \rightarrow X$ , we obtain the augmented simplicial space

$$Y^\bullet = \left( \cdots Y^{[3]} \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \\ \xrightarrow{\pi_3} \end{array} Y^{[2]} \xrightarrow{\pi_1} Y \xrightarrow{\pi} Y^{[0]} \right) \quad (298)$$

of fiberwise cartesian powers of  $Y$ ,  $Y^{[n]} := \underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n \text{ factors}}$ , with  $Y^{[0]} := X$ . The double complex of differential forms

$$\Omega^\bullet(Y^\bullet) = \bigoplus_{n \in \mathbb{N}} \Omega^n(Y^\bullet) = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\substack{r, s \in \mathbb{N} \\ r+s=n}} \Omega^r(Y^{[s]}) \quad (299)$$

on  $Y^\bullet$  has the differential  $d \pm \delta$  coming from the deRham differential  $d$  and the alternating pullback operation

$$\begin{aligned}
\delta : \Omega^r(Y^{[s]}) &\rightarrow \Omega^r(Y^{[s+1]}) \\
\delta : \omega &\mapsto \pi_1^* \omega - \pi_2^* \omega + \pi_3^* \omega + \cdots - (-1)^{s+1} .
\end{aligned} \quad (300)$$

Here we take 0-forms to be valued in  $\mathbb{R}/\mathbb{Z}$ . The map  $\Omega^0(Y) \xrightarrow{d} \Omega^1(Y)$  takes any  $\mathbb{R}$ -valued representative  $f$  of an  $\mathbb{R}/\mathbb{Z}$ -valued form and sends that to the ordinary df. This operation is often denoted

$\Omega^0(Y) \xrightarrow{d\log} \Omega^1(Y)$ . Writing  $\Omega_k^\bullet(Y^\bullet)$  for the space of forms that vanish on  $Y^{[l]}$  for  $l < k$  we define (everything with respect to  $Y$ ):

- A **Deligne  $n$ -cocycle** is a closed element in  $\Omega^n(Y^\bullet)$ ;
- a **flat Deligne  $n$ -cocycle** is a closed element in  $\Omega_1^n(Y^\bullet)$ ;
- a **Deligne coboundary** is an element in  $(d \pm \delta)\Omega_1^\bullet(Y^\bullet)$  (i.e. no component in  $Y^{[0]} = X$ );
- a **shift of connection** is an element in  $(d \pm \delta)\Omega^\bullet(Y^\bullet)$  (i.e. with possibly a contribution in  $Y^{[0]} = X$ ).

The 0-form part of a Deligne cocycle is like the transition function of a  $U(1)$ -bundle. Restricting to this part yields a group homomorphism

$$[\cdot] : H^n(\Omega^\bullet(Y^\bullet)) \longrightarrow H^n(X, \mathbb{Z}) \quad (301)$$

to the integral cohomology on  $X$ . (Notice that the degree on the right is indeed as given, using the total degree on the double complexes  $\Omega^\bullet(Y^\bullet)$  as given.)

Addition of a Deligne coboundary is a gauge transformation. Using the fact [62] that the “fundamental complex”

$$\Omega^r(X) \xrightarrow{\delta} \Omega^r(Y) \xrightarrow{\delta} \Omega^r(Y^{[2]}) \dots \quad (302)$$

is exact for all  $r \geq 1$ , one sees that Deligne cocycles with the same class in  $H^n(X, \mathbb{Z})$  differ by elements in  $(d \pm \delta)\Omega^\bullet(Y^\bullet)$ . Notice that they do not, in general, differ by an element in  $\Omega_1^\bullet(Y^\bullet)$ : two Deligne cochains which differ by an element in  $(d \pm \delta)\Omega_1^\bullet(Y^\bullet)$  describe equivalent line  $n$ -bundles *with equivalent connections*, while those that differ by something in  $(d \pm \delta)\Omega_0^\bullet(Y^\bullet)$  describe equivalent line  $n$ -bundles with possibly inequivalent connections on them.

Let

$$v : \Omega^\bullet(Y^\bullet) \rightarrow \Omega_{\text{vert}}^\bullet(Y) \quad (303)$$

be the map which sends each Deligne  $n$ -cochain  $a$  with respect to  $Y$  to the vertical part of its  $(n-1)$ -form on  $Y^{[1]}$

$$\nu : a \mapsto a|_{\Omega_{\text{vert}}^{n-1}(Y^{[1]})}. \quad (304)$$

(Recall the definition 10 of  $\Omega_{\text{vert}}^\bullet(Y)$ .) Then we have

**Proposition 29** *If two Deligne  $n$ -cocycles  $a$  and  $b$  over  $Y$  have the same class in  $H^n(X, \mathbb{Z})$ , then the classes of  $\nu(a)$  and  $\nu(b)$  coincide.*

*Proof.* As mentioned above,  $a$  and  $b$  have the same class in  $H^n(X, \mathbb{Z})$  if and only if they differ by an element in  $(d \pm \delta)(\Omega^\bullet(Y^\bullet))$ . This means that on  $Y^{[1]}$  they differ by an element of the form

$$d\alpha + \delta\beta = d\alpha + \pi^*\beta. \quad (305)$$

Since  $\pi^*\beta$  is horizontal, this is exact in  $\Omega_{\text{vert}}^\bullet(Y^{[1]})$ .  $\square$

**Proposition 30** *If the  $(n-1)$ -form parts  $B, B' \in \Omega^{n-1}(Y)$  of two Deligne  $n$ -cocycles differ by a  $d \pm \delta$ -exact part, then the two Deligne cocycles have the same class in  $H^n(X, \mathbb{Z})$ .*

*Proof.* If the surjective submersion is not yet contractible, we pull everything back to a contractible refinement, as described in 5.3.1. So assume without restriction of generality that all  $Y^{[n]}$  are contractible. This implies that  $H_{\text{deRham}}^\bullet(Y^{[n]}) = H^0(Y^{[n]})$ , which is a vector space spanned by the connected components of  $Y^{[n]}$ . Now assume

$$B - B' = d\beta + \delta\alpha \quad (306)$$

on  $Y$ . We can immediately see that this implies that the real classes in  $H^n(X, \mathbb{R})$  coincide: the Deligne cocycle property says

$$d(B - B') = \delta(H - H') \quad (307)$$

hence, by the exactness of the deRham complex we have now,

$$\delta(H - H') = \delta(d\alpha) \quad (308)$$

and by the exactness of  $\delta$  we get  $[H] = [H']$ .

To see that also the integral classes coincide we use induction over  $k$  in  $Y^{[k]}$ . For instance on  $Y^{[2]}$  we have

$$\delta(B - B') = d(A - A') \quad (309)$$

and hence

$$\delta d\beta = d(A - A'). \quad (310)$$

Now using again the exactness of the deRham differential  $d$  this implies

$$A - A' = \delta\beta + d\gamma. \quad (311)$$

This way we work our way up to  $Y^{[n]}$ , where it then follows that the 0-form cocycles are coboundant, hence that they have the same class in  $H^n(X, \mathbb{Z})$ .  $\square$

**Proposition 31**  *$b^{n-1}\mathbf{u}(1)$ -descent objects with respect to a given surjective submersion  $Y$  are in bijection with closed vertical  $n$ -forms on  $Y$ :*

$$\left\{ \Omega_{\text{vert}}^\bullet(Y) \xleftarrow{A_{\text{vert}}} \text{CE}(b^{n-1}\mathbf{u}(1)) \right\} \leftrightarrow \{ A_{\text{vert}} \in \Omega_{\text{vert}}^n(Y), dA_{\text{vert}} = 0 \}. \quad (312)$$

*Two such  $b^{n-1}\mathbf{u}(1)$  descent objects on  $Y$  are equivalent precisely if these forms represent the same cohomology class*

$$(A_{\text{vert}} \sim A'_{\text{vert}}) \Leftrightarrow [A_{\text{vert}}] = [A'_{\text{vert}}] \in H^n(\Omega_{\text{vert}}^\bullet(Y)). \quad (313)$$

*Proof.* The first statement is a direct consequence of the definition of  $b^{n-1}\mathbf{u}(1)$  in 6.1. The second statement follows from proposition 10 using the reasoning as in proposition 27.  $\square$

Hence two Deligne cocycles with the same class in  $H^n(X, \mathbb{Z})$  indeed specify the same class of  $b^{n-1}\mathbf{u}(1)$ -descent data.

## 7.2 Connections on $\mathfrak{g}$ -bundles: the extension problem

It turns out that a useful way to conceive of the curvature on a non-flat  $\mathfrak{g}$   $n$ -bundle is, essentially, as the  $(n+1)$ -bundle with connection obstructing the existence of a flat connection on the original  $\mathfrak{g}$ -bundle. This superficially trivial statement is crucial for our way of coming to grips with non-flat higher bundles with connection.

**Definition 33 (descent object for  $\mathfrak{g}$ -connection)** *Given  $\mathfrak{g}$ -bundle descent object*

$$\Omega_{\text{vert}}^\bullet(Y) \xleftarrow{A_{\text{vert}}} \text{CE}(\mathfrak{g}) \quad (314)$$

as above, a  $\mathfrak{g}$ -connection on it is a completion of this morphism to a diagram

$$\begin{array}{ccc}
 \Omega_{\text{vert}}^{\bullet}(Y) & \xleftarrow{A_{\text{vert}}} & \text{CE}(\mathfrak{g}) \\
 \uparrow i^* & & \uparrow \\
 \Omega^{\bullet}(Y) & \xleftarrow{(A, F_A)} & W(\mathfrak{g}) \\
 \uparrow \pi^* & & \uparrow \\
 \Omega^{\bullet}(X) & \xleftarrow{\{K_i\}} & \text{inv}(\mathfrak{g})
 \end{array} \quad (315)$$

As before, two  $\mathfrak{g}$ -connection descent objects are taken to be equivalent, if their pullbacks to a common refinement are concordant.

The top square can always be completed: any representative  $A \in \Omega^{\bullet}(Y)$  of  $A_{\text{vert}} \in \Omega_{\text{vert}}^{\bullet}(Y)$  will do. The curvature  $F_A$  is then uniquely fixed by the dg-algebra homomorphism property. The existence of the top square then says that we have a 1-form on a total space which restricts to a canonical flat 1-form on the fibers. The commutativity of the lower square means that for all invariant polynomials  $P$  of  $\mathfrak{g}$ , the form  $P(F_A)$  on  $Y$  is a form pulled back from  $X$  and is the differential of a form  $cs$  that vanishes on vertical vector fields

$$P(F_A) = \pi^* K. \quad (316)$$

The completion of the bottom square is hence an extra condition: it demands that  $A$  has been chosen such that its curvature  $F_A$  has the property that the form  $P(F_A) \in \Omega^{\bullet}(Y)$  for all invariant polynomials  $P$  are lifted from base space, up to that exact part.

- The commutativity of the top square generalizes the **first Cartan-Ehresmann condition**: the connection form on the total space restricts to a nice form on the fibers.
- The commutativity of the lower square generalizes the **second Cartan-Ehresmann condition**: the connection form on the total space has to behave in such a way that the invariant polynomials applied to its curvature descend down to the base space.

The pullback

$$f^*(Y, (A, F_A)) = (Y', (f^*A, f^*F_A)) \quad (317)$$

of a  $\mathfrak{g}$ -connection descent object  $(Y, (A, F_A))$  on a surjective submersion  $Y$  along a morphism

$$\begin{array}{ccc}
 Y' & \xrightarrow{f} & Y \\
 \searrow \pi' & & \swarrow \pi \\
 & X &
 \end{array} \quad (318)$$

is the  $\mathfrak{g}$ -connection descent object depicted in figure 9.

Notice that the characteristic forms remain unaffected by such a pullback. This way, any two  $\mathfrak{g}$ -connection descent objects may be pulled back to a common surjective submersion. A concordance between two  $\mathfrak{g}$ -connection descent objects on the same surjective submersion is depicted in figure 10.

Suppose  $(A, F_A)$  and  $(A', F_{A'})$  are descent data for  $\mathfrak{g}$ -bundles with connection over the same  $Y$  (possibly after having pulled them back to a common refinement). Then a concordance between them is a diagram as in figure 10.

$$\begin{array}{ccccc}
\Omega_{\text{vert}}^{\bullet}(Y') & \xleftarrow{f^*} & \Omega_{\text{vert}}^{\bullet}(Y) & \xleftarrow{A_{\text{vert}}} & \text{CE}(\mathfrak{g}) \\
\uparrow i'^* & \nwarrow f^* A_{\text{vert}} & \uparrow i^* & & \uparrow \\
\Omega^{\bullet}(Y') & \xleftarrow{f^*} & \Omega^{\bullet}(Y) & \xleftarrow{(A, F_A)} & W(\mathfrak{g}) \\
\uparrow \pi'^* & \nwarrow (f^* A, F_{f^* A}) & \uparrow \pi^* & & \uparrow \\
\Omega^{\bullet}(X) & \xleftarrow{\text{Id}} & \Omega^{\bullet}(X) & \xleftarrow{\{K_i\}} & \text{inv}(\mathfrak{g}) \\
& & \nwarrow \{K_i\} & & 
\end{array}$$

Figure 9: **Pullback of a  $\mathfrak{g}$ -connection descent object**  $(Y, (A, F_A))$  along a morphism  $f : Y' \rightarrow Y$  of surjective submersions, to  $f^*(Y, (A, F_A)) = (Y', (f^* A, F_{f^* A}))$ .

**Definition 34 (equivalence of  $\mathfrak{g}$ -connections)** *We say that two  $\mathfrak{g}$ -connection descent objects are equivalent as  $\mathfrak{g}$ -connection descent objects if they are connected by a vertical concordance namely one for which the derivation part of  $\eta^*$  (according to proposition 10) vanishes on the shifted copy  $\mathfrak{g}^*[1] \hookrightarrow W(\mathfrak{g})$ .*

We have have a closer look at concordance and equivalence of  $\mathfrak{g}$ -connection descent objects in 7.3.

### 7.2.1 Examples.

**Example (ordinary Cartan-Ehresmann connection).** Let  $P \rightarrow X$  be a principal  $G$ -bundle and consider the descent object obtained by setting  $Y = P$  and letting  $A_{\text{vert}}$  be the canonical invariant vertical flat 1-form on fibers  $P$ . Then finding the morphism

$$\Omega^{\bullet}(Y) \xleftarrow{(A, F_A)} W(\mathfrak{g}) \quad (319)$$

such that the top square commutes amounts to finding a 1-form on the total space of the bundle which restricts to the canonical 1-form on the fibers. This is the first of the two conditions on a Cartan-Ehresmann connection. Then requiring the lower square to commute implies requiring that the  $2n$ -forms  $P_i(F_A)$ , formed from the curvature 2-form  $F_A$  and the degree  $n$ -invariant polynomials  $P_i$  of  $\mathfrak{g}$ , have to descend to  $2n$ -forms  $K_i$  on the base  $X$ . But that is precisely the case when  $P_i(F_A)$  is invariant under flows along vertical vector fields. Hence it is true when  $A$  satisfies the second condition of a Cartan-Ehresmann connection, the one that says that the connection form transforms nicely under vertical flows.

**Further examples** appear in 8.3.1.

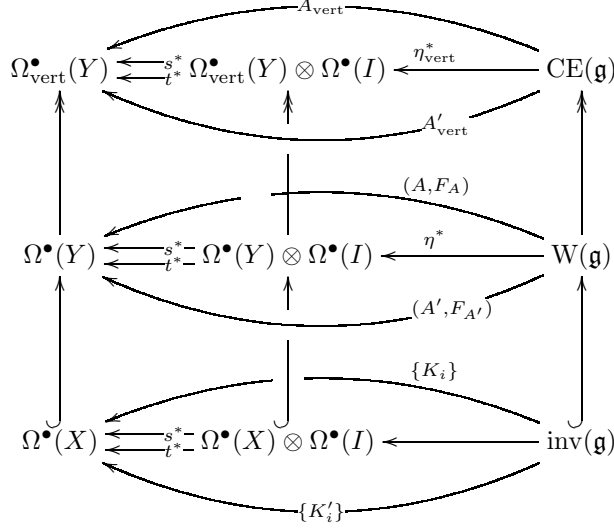


Figure 10: **Concordance between  $\mathfrak{g}$ -connection descent objects**  $(Y, (A, F_A))$  and  $(Y, (A', F_{A'}))$  defined on the same surjective submersion  $\pi : Y \rightarrow X$ . Concordance between descent objects not on the same surjective submersion is reduced to this case by pulling both back to a common refinement, as in figure 9.

### 7.3 Characteristic forms and characteristic classes

**Definition 35** For any  $\mathfrak{g}$ -connection descent object  $(Y, (A, F_A))$  we say that in

$$\begin{array}{ccc}
 \Omega_{\text{vert}}^\bullet(Y) & \xleftarrow{A_{\text{vert}}} & \text{CE}(\mathfrak{g}) \\
 \uparrow i^* & & \uparrow \\
 \Omega^\bullet(Y) & \xleftarrow{(A, F_A)} & W(\mathfrak{g}) \\
 \uparrow \pi^* & & \uparrow \\
 \Omega^\bullet(X) & \xleftarrow{\{K_i\}} & \text{inv}(\mathfrak{g}) \\
 \\ 
 H_{\text{dR}}^\bullet(X) & \xleftarrow{\{[K_i]\}} & H^\bullet(\text{inv}(\mathfrak{g}))
 \end{array} \tag{320}$$

the  $\{K_i\}$  are the characteristic forms, while their deRham classes  $[K_i] \in H_{\text{deRham}}^\bullet(X)$  are the characteristic classes of  $(Y, (A, F_A))$ .

**Proposition 32** If two  $\mathfrak{g}$ -connection descent objects  $(Y, (A, F_A))$  and  $(Y', (A', F_{A'}))$  are related by a concordance as in figure 9 and figure 10 then they have the same characteristic classes:

$$(Y, (A, F_A)) \sim (Y', (A', F_{A'})) \Rightarrow \{[K_i]\} = \{[K'_i]\}. \tag{321}$$

*Proof.* We have seen that pullback does not change the characteristic forms. It follows from proposition 27 that the characteristic classes are invariant under concordance.  $\square$

**Corollary 2** If two  $\mathfrak{g}$ -connection descent objects are equivalent according to definition 34, then they even have the same characteristic forms.



Proof. For concordances between equivalent  $\mathfrak{g}$ -connection descent objects the derivation part of  $\eta^*$  is vertical and therefore vanishes on  $\text{inv}(\mathfrak{g}) = W(\mathfrak{g})_{\text{basic}}$ .  $\square$

**Remark. (shifts of  $\mathfrak{g}$ -connections)** We observe that, by the very definition of  $W(\mathfrak{g})$ , any shift in the connection  $A$ ,

$$A \mapsto A' = A + D \in \Omega^\bullet(Y, \mathfrak{g})$$

can be understood as a transformation

$$\begin{array}{ccc} & (A, F_A) & \\ \swarrow & \Downarrow \rho & \searrow \\ \Omega^\bullet(Y) & & W(\mathfrak{g}) \\ \nwarrow & \Downarrow \rho & \nearrow \\ & (A+D, F_{A+D}) & \end{array}$$

with the property that  $\rho$  vanishes on the *non*-shifted copy  $\mathfrak{g}^* \hookrightarrow W(\mathfrak{g})$  and is nontrivial only on the shifted copy  $\mathfrak{g}^*[1] \hookrightarrow W(\mathfrak{g})$ : in that case for  $a \in \mathfrak{g}^*$  any element in the unshifted copy, we have

$$(A + D)(a) = A(a) + [d, \rho](a) = A(a) + \rho(d_{W(\mathfrak{g})}a) = A(a) + \rho(d_{CE(\mathfrak{g})}a + \sigma a) = A(a) + \rho(\sigma a).$$

and hence  $D(a) = \rho(\sigma)$ , which uniquely fixes  $\rho$  in terms of  $D$  and vice versa.

Therefore concordances which are not purely vertical describe homotopies between  $\mathfrak{g}$ -connection descent objects in which the connection is allowed to vary.

**Remark (gauge transformations versus shifts of the connections).** We therefore obtain the following picture.

- Vertical concordances relate gauge equivalent  $\mathfrak{g}$ -connections (compare definition 29 of gauge transformations of  $\mathfrak{g}$ -valued forms)

$$\begin{array}{ccccc} & & \Omega_{\text{vert}}^\bullet(Y) & \xleftarrow{s^*} \Omega_{\text{vert}}^\bullet(Y) \otimes \Omega^\bullet(I) & \xleftarrow{\eta_{\text{vert}}^*} CE(\mathfrak{g}) \\ & & \uparrow & \uparrow & \uparrow \\ & & \Omega^\bullet(Y) & \xleftarrow{s^*} \Omega^\bullet(Y) \otimes \Omega^\bullet(I) & \xleftarrow{\eta^*} W(\mathfrak{g}) \\ & & \uparrow & \uparrow & \uparrow \\ & & \Omega^\bullet(X) & \xleftarrow{s^*} \Omega^\bullet(X) \otimes \Omega^\bullet(I) & \xleftarrow{} \text{inv}(\mathfrak{g}) \end{array} \quad \begin{array}{l} \xleftarrow{A_{\text{vert}}} \\ \xleftarrow{A'_{\text{vert}}} \\ \xleftarrow{(A, F_A)} \\ \xleftarrow{(A', F_{A'})} \end{array} \quad \begin{array}{l} \Rightarrow \\ \exists \end{array} \quad \begin{array}{l} (A, F_A) \text{ and } (A', F_{A'}) \text{ are} \\ \text{equivalent as } \mathfrak{g}\text{-connections} \end{array} \quad ; \quad (322)$$

- Non-vertical concordances relate  $\mathfrak{g}$ -connection descent objects whose underlying  $\mathfrak{g}$ -descent object – the underlying  $\mathfrak{g}$ - $n$ -bundles – are equivalent, but which possibly differ in the choice of connection on these

$\mathfrak{g}$ -bundles:

$$\begin{array}{ccccc}
& & \Omega_{\text{vert}}^{\bullet}(Y) & \xleftarrow[s_t^*]{s^*} \Omega_{\text{vert}}^{\bullet}(Y) \otimes \Omega^{\bullet}(I) & \xleftarrow{\eta_{\text{vert}}^*} \text{CE}(\mathfrak{g}) \\
& & \uparrow & \uparrow & \uparrow \\
(A', F_{A'}) \text{ is obtained from } & \Leftrightarrow \exists & \Omega^{\bullet}(Y) & \xleftarrow[s_t^*]{s^*} \Omega^{\bullet}(Y) \otimes \Omega^{\bullet}(I) & \xleftarrow{\eta^*} W(\mathfrak{g}) \\
(A, F_A) \text{ by a } & & \uparrow & \uparrow & \uparrow \\
\text{shift of } \mathfrak{g}\text{-connections} & & \Omega^{\bullet}(X) & \xleftarrow[s_t^*]{s^*} \Omega^{\bullet}(X) \otimes \Omega^{\bullet}(I) & \xleftarrow{} \text{inv}(\mathfrak{g}) \\
& & \uparrow & \uparrow & \uparrow \\
& & \Omega^{\bullet}(X) & \xleftarrow[s_t^*]{s^*} \Omega^{\bullet}(X) \otimes \Omega^{\bullet}(I) & \xleftarrow{} \text{inv}(\mathfrak{g})
\end{array}
\quad (323)$$

$\{K_i\}$  (curved arrow from  $\Omega^{\bullet}(X)$  to  $\Omega^{\bullet}(Y)$ )  
 $\{K'_i\}$  (curved arrow from  $\Omega^{\bullet}(X)$  to  $\Omega^{\bullet}(Y)$ )  
 $(A, F_A)$  (curved arrow from  $\Omega^{\bullet}(Y)$  to  $W(\mathfrak{g})$ )  
 $(A', F_{A'})$  (curved arrow from  $\Omega^{\bullet}(Y)$  to  $W(\mathfrak{g})$ )

**Remark.** This in particular shows that for a given  $\mathfrak{g}$ -descent object

$$\Omega_{\text{vert}}^{\bullet}(Y) \xleftarrow{A_{\text{vert}}} \text{CE}(\mathfrak{g})$$

the corresponding characteristic classes obtained by choosing a connection  $(A, F_A)$  does not depend on continuous variations of that choice of connection.

In the case of ordinary Lie (1-)algebras  $\mathfrak{g}$  it is well known that any two connections on the same bundle may be continuously connected by a path of connections: the space of 1-connections is an affine space modeled on  $\Omega^1(X, \mathfrak{g})$ . If we had an analogous statement for  $\mathfrak{g}$ -connections for higher  $L_{\infty}$ -algebras, we could strengthen the above statement.

### 7.3.1 Examples

**Ordinary characteristic classes of  $\mathfrak{g}$ -bundles** Let  $\mathfrak{g}$  be an ordinary Lie algebra and  $(Y, (A, F_A))$  be a  $\mathfrak{g}$ -descent object corresponding to an ordinary Cartan-Ehresmann connection as in 7.2.1. Using the fact 13 we know that  $\text{inv}(\mathfrak{g})$  contains all the ordinary invariant polynomials  $P$  on  $\mathfrak{g}$ . Hence the characteristic classes  $[P(F_A)]$  are precisely the standard characteristic classes (in deRham cohomology) of the  $G$ -bundle with connection.

**Characteristic classes of  $b^{n-1}\mathfrak{u}(1)$ -bundles.-** For  $\mathfrak{g} = b^{n-1}\mathfrak{u}(1)$  we have, according to proposition 17,  $\text{inv}(b^{n-1}\mathfrak{u}(1)) = \text{CE}(b^n\mathfrak{u}(1))$  and hence a single degree  $n + 1$  characteristic class: the curvature itself.

This case we had already discussed in the context of Deligne cohomology in 7.1.1. In particular, notice that in definition 32 we had already encountered the distinction between homotopies of  $L_{\infty}$ -algebra that are or are not pure gauge transformations, in that they do or do not shift the connection: what is called a *Deligne coboundary* in definition 32 corresponds to an equivalence of  $b^{n-1}\mathfrak{u}$ -connection descent objects as in 322, while what is called a *shift of connection* there corresponds to a concordance that involves a shift as in 323.

## 7.4 Universal and generalized $\mathfrak{g}$ -connections

We can generalize the discussion of  $\mathfrak{g}$ -bundles with connection on spaces  $X$ , by

- allowing all occurrences of the algebra of differential forms to be replaced with more general differential graded algebras; this amounts to admitting generalized smooth spaces as in 5.1;

- by allowing all Chevalley-Eilenberg and Weil algebras of  $L_\infty$ -algebras to be replaced by DGCAs which may be nontrivial in degree 0. This amounts to allowing not just structure  $\infty$ -groups but also structure  $\infty$ -groupoids.

**Definition 36 (generalized  $\mathfrak{g}$ -connection descent objects)** *Given any  $L_\infty$ -algebra  $\mathfrak{g}$ , and given any DGCA  $A$ , we say a  $\mathfrak{g}$ -connection descent object for  $A$  is*

- a surjection  $F \xleftarrow{i^*} P$  such that  $A \simeq P_{\text{basic}}$ ;
- a choice of horizontal morphisms in the diagram

$$\begin{array}{ccc}
 F & \xleftarrow{A_{\text{vert}}} & \text{CE}(\mathfrak{g}) \\
 \uparrow i^* & & \uparrow \\
 P & \xleftarrow{(A, F_A)} & W(\mathfrak{g}) \\
 \uparrow & & \uparrow \\
 A & \xleftarrow{\{K_i\}} & \text{inv}(\mathfrak{g})
 \end{array} ; \tag{324}$$

The notion of equivalence of these descent objects is as before.

**Horizontal forms** Given any algebra surjection

$$\begin{array}{c}
 F \\
 \uparrow i^* \\
 P
 \end{array}$$

we know from definition 8 what the “vertical directions” on  $P$  are. After we have chosen a  $\mathfrak{g}$ -connection on  $P$ , we obtain also notion of *horizontal* elements in  $P$ :

**Definition 37 (horizontal elements)** *Given a  $\mathfrak{g}$ -connection  $(A, F_A)$  on  $P$ , the algebra of horizontal elements*

$$\text{hor}_A(P) \subset P$$

*of  $P$  with respect to this connection are those elements not in the ideal generated by the image of  $A$ .*

Notice that  $\text{hor}_A(P)$  is in general just a graded-commutative algebra, not a differential algebra. Accordingly the inclusion  $\text{hor}_A(P) \subset P$  is meant just as an inclusion of algebras.

$$\begin{array}{ccc}
\mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mathrm{Id}} & \mathrm{CE}(\mathfrak{g}) \\
\uparrow i^* & & \uparrow \\
W(\mathfrak{g}) & \xleftarrow{\mathrm{Id}} & W(\mathfrak{g}) \\
\uparrow & & \uparrow \\
\mathrm{inv}(\mathfrak{g}) & \xleftarrow{\mathrm{Id}} & \mathrm{inv}(\mathfrak{g})
\end{array}
\tag{326}$$

Figure 11: **The universal  $\mathfrak{g}$ -connection** descent object.

#### 7.4.1 Examples.

**The universal  $\mathfrak{g}$ -connection.** The tautological example is actually of interest: for any  $L_\infty$ -algebra  $\mathfrak{g}$ , there is a *canonical*  $\mathfrak{g}$ -connection descent object on  $\mathrm{inv}(\mathfrak{g})$ . This comes from choosing

$$( F \xleftarrow{i^*} P ) := ( \mathrm{CE}(\mathfrak{g}) \xleftarrow{i^*} W(\mathfrak{g}) ) \tag{325}$$

and then taking the horizontal morphisms to be all identities, as shown in figure 11:

We can then finally give an *intrinsic* interpretation of the decomposition of the generators of the Weil algebra  $W(\mathfrak{g})$  of any  $L_\infty$ -algebra into elements in  $\mathfrak{g}^*$  and elements in the shifted copy  $\mathfrak{g}^*[1]$ , which is crucial for various of our constructions (for instance for the vanishing condition in 149):

**Proposition 33** *The horizontal elements of  $W(\mathfrak{g})$  with respect to the universal  $\mathfrak{g}$ -connection  $(A, F_A)$  on  $W(\mathfrak{g})$  are precisely those generated entirely from the shifted copy  $\mathfrak{g}^*[1]$ :*

$$\mathrm{hor}_A(W(\mathfrak{g})) = \wedge^\bullet(\mathfrak{g}^*[1]) \subset W(\mathfrak{g}) .$$

#### Line $n$ -bundles on classifying spaces

**Proposition 34** *Let  $\mathfrak{g}$  be any  $L_\infty$ -algebra and  $P \in \mathrm{inv}(\mathfrak{g})$  any closed invariant polynomial on  $\mathfrak{g}$  of degree  $n+1$ . Let  $\mathrm{cs} := \tau P$  be the transgression element and  $\mu := i^* \mathrm{cs}$  the cocycle that  $P$  transgresses to according to proposition 15. Then we canonically obtain a  $b^{n-1}\mathfrak{u}(1)$ -connection descent object in  $\mathrm{inv}(\mathfrak{g})$ :*

$$\begin{array}{ccc}
\mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \\
\uparrow i^* & & \uparrow \\
W(\mathfrak{g}) & \xleftarrow{(\mathrm{cs}, P)} & W(b^{n-1}\mathfrak{u}(1)) \\
\uparrow & & \uparrow \\
\mathrm{inv}(\mathfrak{g}) & \xleftarrow{P} & \mathrm{inv}(b^n\mathfrak{u}(1)) = \mathrm{CE}(b^{n-1}\mathfrak{u}(1))
\end{array}
\tag{327}$$

**Remark.** For instance for  $\mathfrak{g}$  an ordinary semisimple Lie algebra and  $\mu$  its canonical 3-cocycle, we obtain a descent object for a Lie 3-bundle which plays the role of what is known as the canonical 2-gerbe on the classifying space  $BG$  of the simply connected group  $G$  integrating  $\mathfrak{g}$  [21]. From the above and using 6.5.1 we read off that its connection 3-form is the canonical Chern-Simons 3-form. We will see this again in 9.3.1, where we show that the 3-particle (the 2-brane) coupled to the above  $\mathfrak{g}$ -connection descent object indeed reproduces Chern-Simons theory.

## 8 Higher String- and Chern-Simons $n$ -bundles: the lifting problem

We discuss the general concept of weak cokernels of morphisms of  $L_\infty$ -algebras. Then we apply this to the special problem of lifts of differential  $\mathfrak{g}$ -cocycles through String-like extensions.

### 8.1 Weak cokernels of $L_\infty$ -morphisms

After introducing the notion of a mapping cone of qDGCAs, the main point here is proposition 40, which establishes the existence of the weak inverse  $f^{-1}$  that was mentioned in 2.3. It will turn out to be that very weak inverse which picks up the information about the existence or non-existence of the lifts discussed in 8.3. We can define the weak cokernel for *normal*  $L_\infty$ -subalgebras:

**Definition 38 (normal  $L_\infty$ -subalgebra)** We say a Lie  $\infty$ -algebra  $\mathfrak{h}$  is a normal sub  $L_\infty$ -algebra of the  $L_\infty$ -algebra  $\mathfrak{g}$  if there is a morphism

$$\mathrm{CE}(\mathfrak{h}) \xleftarrow{t^*} \mathrm{CE}(\mathfrak{g}) \quad (328)$$

which the property that

- on  $\mathfrak{g}^*$  it restricts to a surjective linear map  $\mathfrak{h}^* \xleftarrow{t_1^*} \mathfrak{g}^*$  ;
- if  $a \in \ker(t^*)$  then  $d_{\mathrm{CE}(\mathfrak{g})}a \in \wedge^\bullet(\ker(t_1^*))$ .

**Proposition 35** For  $\mathfrak{h}$  and  $\mathfrak{g}$  ordinary Lie algebras, the above notion of normal sub  $L_\infty$ -algebra coincides with the standard notion of normal sub Lie algebras.

*Proof.* If  $a \in \ker(t^*)$  then for any  $x, y \in \mathfrak{g}$  the condition says that  $(d_{\mathrm{CE}(\mathfrak{g})}a)(x \vee y) = -a(D[x \vee y]) = -a([x, y])$  vanishes when  $x$  or  $y$  are in the image of  $t$ . But  $a([x, y])$  vanishes when  $[x, y]$  is in the image of  $t$ . Hence the condition says that if at least one of  $x$  and  $y$  is in the image of  $t$ , then their bracket is.  $\square$

**Definition 39 (mapping cone of qDGCAs; crossed module of normal sub  $L_\infty$ -algebras)** Let  $t : \mathfrak{h} \hookrightarrow \mathfrak{g}$  be an inclusion of a normal sub  $L_\infty$ -algebra  $\mathfrak{h}$  into  $\mathfrak{g}$ . The mapping cone of  $t^*$  is the qDGCA whose underlying graded algebra is

$$\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1]) \quad (329)$$

and whose differential  $d_t$  is such that it acts on generators schematically as

$$d_t = \begin{pmatrix} d_{\mathfrak{g}} & 0 \\ t^* & d_{\mathfrak{h}} \end{pmatrix}. \quad (330)$$

In more detail,  $d_{t^*}$  is defined as follows. We write  $\sigma t^*$  for the degree +1 derivation on  $\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1])$  which acts on  $\mathfrak{g}^*$  as  $t^*$  followed by a shift in degree and which acts on  $\mathfrak{h}^*[1]$  as 0. Then, for any  $a \in \mathfrak{g}^*$ , we have

$$d_t a := d_{\text{CE}(\mathfrak{g})} a + \sigma t^*(a). \quad (331)$$

and

$$d_t \sigma t^*(a) := -\sigma t^*(d_{\text{CE}(\mathfrak{g})} a) = -d_t d_{\text{CE}(\mathfrak{g})} a. \quad (332)$$

Notice that the last equation

- defines  $d_t$  on all of  $\mathfrak{h}^*[1]$  since  $t^*$  is surjective;
- is well defined in that it agrees for  $a$  and  $a'$  if  $t^*(a) = t^*(a')$ , since  $t$  is normal.

**Proposition 36** *The differential  $d_t$  defined this way indeed satisfies  $(d_t)^2 = 0$ .*

Proof. For  $a \in \mathfrak{g}^*$  we have

$$d_t d_t a = d_t(d_{\text{CE}(\mathfrak{g})} a + \sigma t^*(a)) = \sigma t^*(d_{\text{CE}(\mathfrak{g})} a) - \sigma t^*(d_{\text{CE}(\mathfrak{g})} a) = 0. \quad (333)$$

Hence  $(d_t)^2$  vanishes on  $\wedge^\bullet(\mathfrak{g}^*)$ . Since

$$d_t d_t \sigma t^*(a) = -d_t d_t d_{\text{CE}(\mathfrak{g})} a \quad (334)$$

and since  $d_{\text{CE}(\mathfrak{g})} a \in \wedge^\bullet(\mathfrak{g}^*)$  this implies  $(d_t)^2 = 0$ .  $\square$

We write  $\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) := (\wedge^\bullet(\mathfrak{g}^* \oplus \mathfrak{h}^*[1]), d_t)$  for the resulting qDGCA and  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  for the corresponding  $L_\infty$ -algebra.

The next proposition asserts that  $\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  is indeed a (weak) kernel of  $t^*$ .

**Proposition 37** *There is a canonical morphism  $\text{CE}(\mathfrak{g}) \longleftarrow \text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  with the property that*

$$\begin{array}{ccccc} \text{CE}(\mathfrak{h}) & \xleftarrow{t^*} & \text{CE}(\mathfrak{g}) & \xleftarrow{\quad} & \text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \\ & \searrow & \uparrow \tau & \nearrow & \\ & & 0 & & \end{array} \quad (335)$$

Proof. On components, this morphism is the identity on  $\mathfrak{g}^*$  and 0 on  $\mathfrak{h}^*[1]$ . One checks that this respects the differentials. The homotopy to the 0-morphism sends

$$\tau : \sigma t^*(a) \mapsto t^*(a). \quad (336)$$

Using definition 17 one checks that then indeed

$$[d, \tau] : a \mapsto \tau(d_{\text{CE}(\mathfrak{g})} a + \sigma t^* a) = a \quad (337)$$

and

$$[d, \tau] : \sigma t^* a \mapsto d_{\text{CE}(\mathfrak{g})} a + \tau(-\sigma t^*(d_{\text{CE}(\mathfrak{g})} a)) = 0. \quad (338)$$

Here the last step makes crucial use of the condition 149 which demands that

$$\tau(d_{\text{W}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} \sigma t^* a - d_{\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})} \sigma t^* a) = 0 \quad (339)$$

and the formula (145) which induces precisely the right combinatorial factors.  $\square$

Notice that not only is  $\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  in the kernel of  $t^*$ , it is indeed the universal object with this property, hence is *the* kernel of  $t^*$  (of course up to equivalence).

**Proposition 38** *Let  $\text{CE}(\mathfrak{h}) \xleftarrow{t^*} \text{CE}(\mathfrak{g}) \xleftarrow{u^*} \text{CE}(\mathfrak{f})$  be a sequence of  $q\text{DGCA}$ s with  $t^*$  normal, as above, and with the property that  $u^*$  restricts, on the underlying vector spaces of generators, to the kernel of the linear map underlying  $t^*$ . Then there is a unique morphism  $f : \text{CE}(\mathfrak{f}) \rightarrow \text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  such that*

$$\begin{array}{ccccc} \text{CE}(\mathfrak{h}) & \xleftarrow{t^*} & \text{CE}(\mathfrak{g}) & \xleftarrow{\quad} & \text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \\ & & \uparrow u^* & \nearrow f & \\ & & \text{CE}(\mathfrak{f}) & & \end{array} \quad (340)$$

*Proof.* The morphism  $f$  has to be in components the same as  $\text{CE}(\mathfrak{g}) \leftarrow \text{CE}(\mathfrak{f})$ . By the assumption that this is in the kernel of  $t^*$ , the differentials are respected.  $\square$

**Remark.** There should be a generalization of the entire discussion where  $u^*$  is not restricted to be the kernel of  $t^*$  on generators. However, for our application here, this simple situation is all we need.

**Proposition 39** *For a string-like extension  $\mathfrak{g}_\mu$  from definition 23, the morphism*

$$\text{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{t^*} \text{CE}(\mathfrak{g}_\mu) \quad (341)$$

*is normal in the sense of definition 38.*

**Proposition 40** *In the case that the sequence*

$$\text{CE}(\mathfrak{h}) \xleftarrow{t^*} \text{CE}(\mathfrak{g}) \xleftarrow{u^*} \text{CE}(\mathfrak{f}) \quad (342)$$

*above is a String-like extension*

$$\text{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{t^*} \text{CE}(\mathfrak{g}_\mu) \xleftarrow{u^*} \text{CE}(\mathfrak{g}) \quad (343)$$

*from proposition 20 or the corresponding Weil-algebra version*

$$\begin{array}{ccccc} W(b^{n-1}\mathfrak{u}(1)) & \xleftarrow{t^*} & W(\mathfrak{g}_\mu) & \xleftarrow{u^*} & W(\mathfrak{g}) \\ \downarrow = & & \downarrow = & & \downarrow = \\ \text{CE}(\text{inn}(b^{n-1}\mathfrak{u}(1))) & \xleftarrow{t^*} & \text{CE}(\text{inn}(\mathfrak{g}_\mu)) & \xleftarrow{u^*} & \text{CE}(\text{inn}(\mathfrak{g})) \end{array} \quad (344)$$

*the morphisms  $f : \text{CE}(\mathfrak{f}) \rightarrow \text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  and  $\hat{f} : W(\mathfrak{f}) \rightarrow W(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  have weak inverses  $f^{-1} : \text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \rightarrow \text{CE}(\mathfrak{f})$  and  $\hat{f}^{-1} : W(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \rightarrow W(\mathfrak{f})$ , respectively.*

*Proof.* We first construct a morphism  $f^{-1}$  and then show that it is weakly inverse to  $f$ . The statement for  $\hat{f}$  follows from the functoriality of forming the Weil algebra, proposition 7. Start by choosing a splitting of the vector space  $V$  underlying  $\mathfrak{g}^*$  as

$$V = \ker(t^*) \oplus V_1. \quad (345)$$

This is the non-canonical choice we need to make. Then take the component map of  $f^{-1}$  to be the identity on  $\ker(t^*)$  and 0 on  $V_1$ . Moreover, for  $a \in V_1$  set

$$f^{-1} : \sigma t^*(a) \mapsto -(d_{\text{CE}(\mathfrak{g})} a)|_{\wedge^\bullet \ker(t^*)}, \quad (346)$$

where the restriction is again with respect to the chosen splitting of  $V$ . We check that this assignment, extended as an algebra homomorphism, does respect the differentials.

For  $a \in \ker(t^*)$  we have

$$\begin{array}{ccc} a & \xrightarrow{d_t} & d_{\text{CE}(\mathfrak{g})}a \\ f^{-1} \downarrow & & \downarrow f^{-1} \\ a & \xrightarrow{d_{\text{CE}(\mathfrak{f})}} & d_{\text{CE}(\mathfrak{g})}a \end{array} \quad (347)$$

using the fact that  $t^*$  is normal. For  $a \in V_1$  we have

$$\begin{array}{ccc} a & \xrightarrow{d_t} & d_{\text{CE}(\mathfrak{g})}a + \sigma t^*(a) \\ f^{-1} \downarrow & & \downarrow f^{-1} \\ 0 & \xrightarrow{d_{\text{CE}(\mathfrak{f})}} & (d_{\text{CE}(\mathfrak{g})}a)|_{\wedge^\bullet \ker(t^*)} - (d_{\text{CE}(\mathfrak{g})}a)|_{\wedge^\bullet \ker(t^*)} \end{array} \quad (348)$$

and

$$\begin{array}{ccc} \sigma t^*(a) & \xrightarrow{d_t} & -\sigma t^*(d_{\text{CE}(\mathfrak{g})}a) \\ f^{-1} \downarrow & & \downarrow f^{-1} \\ -(d_{\text{CE}(\mathfrak{g})}a)|_{\wedge^\bullet \ker(t^*)} & \xrightarrow{d_{\text{CE}(\mathfrak{f})}} & -d_{\text{CE}(\mathfrak{f})}((d_{\text{CE}(\mathfrak{g})}a)|_{\ker(t^*)}) \end{array} \quad (349)$$

This last condition happens to be satisfied for the examples stated in the proposition. The details for that are discussed in 8.1.1 below. By the above,  $f^{-1}$  is indeed a morphism of qDGCAs.

Next we check that  $f^{-1}$  is a weak inverse of  $f$ . Clearly

$$\text{CE}(\mathfrak{f}) \longleftarrow \text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \longleftarrow \text{CE}(\mathfrak{f}) \quad (350)$$

is the identity on  $\text{CE}(\mathfrak{f})$ . What remains is to construct a homotopy

$$\begin{array}{ccccc} \text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) & \longleftarrow & \text{CE}(\mathfrak{f}) & \longleftarrow & \text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \\ & & \Downarrow \tau & & \\ & & \text{Id} & & \end{array} \quad (351)$$

One checks that this is accomplished by taking  $\tau$  to act on  $\sigma V_1$  as  $\tau : \sigma V_1 \xrightarrow{\cong} V_1$  and extended suitably.  $\square$

### 8.1.1 Examples

**Weak cokernel for the String-like extension.** Let our sequence

$$\text{CE}(\mathfrak{h}) \xleftarrow{t^*} \text{CE}(\mathfrak{g}) \xleftarrow{u^*} \text{CE}(\mathfrak{f}) \quad (352)$$

be a String-like extension

$$\text{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{t^*} \text{CE}(\mathfrak{g}_\mu) \xleftarrow{u^*} \text{CE}(\mathfrak{g}) \quad (353)$$

from proposition 20. Then the mapping cone Chevalley-Eilenberg algebra

$$\text{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \quad (354)$$



is

$$\wedge^\bullet (\mathfrak{g}^* \oplus \mathbb{R}[n] \oplus \mathbb{R}[n+1]) \quad (355)$$

with differential given by

$$d_t|_{\mathfrak{g}^*} = d_{\text{CE}(\mathfrak{g})} \quad (356)$$

$$d_t|_{\mathbb{R}[n]} = -\mu + \sigma \quad (357)$$

$$d_t|_{\mathbb{R}[n+1]} = 0. \quad (358)$$

(As always,  $\sigma$  is the canonical degree shifting isomorphism on generators extended as a derivation.) The morphism

$$\text{CE}(\mathfrak{g}) \xleftarrow[\simeq]{f^{-1}} \text{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \quad (359)$$

acts as

$$f^{-1}|_{\mathfrak{g}^*} = \text{Id} \quad (360)$$

$$f^{-1}|_{\mathbb{R}[n]} = 0 \quad (361)$$

$$f^{-1}|_{\mathbb{R}[n+1]} = \mu. \quad (362)$$

To check the condition in equation 349 explicitly in this case, let  $b \in \mathbb{R}[n]$  and write  $b := t^*b$  for simplicity (since  $t^*$  is the identity on  $\mathbb{R}[n]$ ). Then

$$\begin{array}{ccc} \sigma b & \xrightarrow{d_t} & 0 \\ \downarrow f^{-1} & & \downarrow f^{-1} \\ \mu & \xrightarrow{d_{\text{CE}(\mathfrak{g})}} & 0 \end{array} \quad (363)$$

does commute.

**Weak cokernel for the String-like extension in terms of the Weil algebra.** We will also need the analogous discussion not for the Chevalley-Eilenberg algebras, but for the corresponding Weil algebras. To that end consider now the sequence

$$\text{W}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{t^*} \text{W}(\mathfrak{g}_\mu) \xleftarrow{u^*} \text{W}(\mathfrak{g}). \quad (364)$$

This is handled most conveniently by inserting the isomorphism

$$\text{W}(\mathfrak{g}_\mu) \simeq \text{CE}(\text{cs}_P(\mathfrak{g})) \quad (365)$$

from proposition 21 as well as the identification

$$\text{W}(\mathfrak{g}) = \text{CE}(\text{inn}(\mathfrak{g})) \quad (366)$$

such that we get

$$\text{CE}(\text{inn}(b^{n-1}\mathfrak{u}(1))) \xleftarrow{t^*} \text{CE}(\text{cs}_P(\mathfrak{g})) \xleftarrow{u^*} \text{CE}(\text{inn}(\mathfrak{g})). \quad (367)$$

Then we find that the mapping cone algebra  $\text{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \text{cs}_P(\mathfrak{g}))$  is

$$\wedge^\bullet (\mathfrak{g}^* \oplus \mathfrak{g}^*[1] \oplus (\mathbb{R}[n] \oplus \mathbb{R}[n+1]) \oplus (\mathbb{R}[n+1] \oplus \mathbb{R}[n+2])). \quad (368)$$

Write  $b$  and  $c$  for the canonical basis elements of  $\mathbb{R}[n] \oplus \mathbb{R}[n+1]$ , then the differential is characterized by

$$d_t|_{\mathfrak{g}^* \oplus \mathfrak{g}^*} = d_{W(\mathfrak{g})} \quad (369)$$

$$d_t : b \mapsto c - cs + \sigma b \quad (370)$$

$$d_t : c \mapsto P + \sigma c \quad (371)$$

$$d_t : \sigma b \mapsto -\sigma c \quad (372)$$

$$d_t : \sigma c \mapsto 0. \quad (373)$$

Notice above the relative sign between  $\sigma b$  and  $\sigma c$ . This implies that the canonical injection

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathrm{cs}_P(\mathfrak{g})) \xleftarrow{i} W(b^n\mathfrak{u}(1)) \quad (374)$$

also carries a sign: if we denote the degree  $n+1$  and  $n+2$  generators of  $W(b^n\mathfrak{u}(1))$  by  $h$  and  $dh$ , then

$$i : h \mapsto \sigma b \quad (375)$$

$$i : dh \mapsto -\sigma c. \quad (376)$$

This sign has no profound structural role, but we need to carefully keep track of it, for instance in order for our examples in 8.3.1 to come out right. The morphism

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathrm{cs}_P(\mathfrak{g})) \xleftarrow[\simeq]{f^{-1}} W(\mathfrak{g}) \quad (377)$$

acts as

$$f^{-1}|_{\mathfrak{g}^* \oplus \mathfrak{g}^*[1]} = \mathrm{Id} \quad (378)$$

$$f^{-1} : \sigma b \mapsto cs \quad (379)$$

$$f^{-1} : \sigma c \mapsto -P. \quad (380)$$

Again, notice the signs, as they follow from the general prescription in proposition 40. We again check explicitly equation (349):

$$\begin{array}{ccc} \sigma b & \xrightarrow{d_t} & -\sigma c \\ \downarrow f^{-1} & & \downarrow f^{-1} \\ cs & \xrightarrow{d_{W(\mathfrak{g})}} & P \end{array} . \quad (381)$$

## 8.2 Lifts of $\mathfrak{g}$ -descent objects through String-like extensions

We need the above general theory for the special case where we have the mapping cone  $\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$  as the weak kernel of the left morphism in a String-like extension

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{\quad} \mathrm{CE}(\mathfrak{g}_\mu) \xleftarrow{\quad} \mathrm{CE}(\mathfrak{g}) \quad (382)$$

coming from an  $(n+1)$  cocycle  $\mu$  on an ordinary Lie algebra  $\mathfrak{g}$ . In this case  $\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu)$  looks like

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) = (\wedge^\bullet(\mathfrak{g}^* \oplus \mathbb{R}[n] \oplus \mathbb{R}[n+1]), d_t). \quad (383)$$

By chasing this through the above definitions, we find

**Proposition 41** *The morphism*

$$f^{-1} : \mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \rightarrow \mathrm{CE}(\mathfrak{g}) \quad (384)$$

vanishes on  $\mathbb{R}[n]$

$$f^{-1}|_{\mathbb{R}[n]} : b \mapsto 0, \quad (386)$$

and satisfies

$$f^{-1}|_{\mathbb{R}[n+1]} : \sigma t^* b \mapsto \mu. \quad (387)$$

Therefore we find the  $(n + 1)$ -cocycle

$$\Omega_{\text{vert}}^{\bullet}(Y) \xleftarrow{\hat{A}_{\text{vert}}} \text{CE}(b^n_{\text{u}}(1)) \quad (388)$$

obstructing the lift of a  $\mathfrak{g}$ -cocycle

$$\Omega_{\text{vert}}^\bullet(Y) \xleftarrow{A_{\text{vert}}} \text{CE}(\mathfrak{g}) \ , \quad (389)$$

according to 2.3 given by

$$\begin{array}{c}
\text{CE}(b^{n-1}u(1) \hookrightarrow \mathfrak{g}_\mu) \xleftarrow{j} \text{CE}(b^n u(1)) \quad (390) \\
\searrow f^{-1} \\
\text{CE}(b^{n-1}u(1)) \xleftarrow{i^*} \text{CE}(\mathfrak{g}_\mu) \xleftarrow{\quad} \text{CE}(\mathfrak{g}) \\
\swarrow \text{A}_{\text{vert}} \quad \searrow \hat{\text{A}}_{\text{vert}} \\
\Omega_{\text{vert}}^\bullet(Y)
\end{array}$$

to be the  $(n + 1)$ -form

$$\mu(A_{\text{vert}}) \in \Omega_{\text{vert}}^{n+1}(Y). \quad (391)$$

**Proposition 42** *Let  $A_{\text{vert}} \in \Omega_{\text{vert}}^1(Y, \mathfrak{g})$  be the cocycle of a  $G$ -bundle  $P \rightarrow X$  for  $\mathfrak{g}$  semisimple and let  $\mu = \langle \cdot, [\cdot, \cdot] \rangle$  be the canonical 3-cocycle. Then  $\mathfrak{g}_\mu$  is the standard String Lie 3-algebra and the obstruction to lifting  $P$  to a String 2-bundle, i.e. lifting to a  $\mathfrak{g}_\mu$ -cocycle, is the Chern-Simons 3-bundle with cocycle given by the vertical 3-form*

$$\langle A_{\text{vert}} \wedge [A_{\text{vert}} \wedge A_{\text{vert}}] \rangle \in \Omega_{\text{vert}}^3(Y). \quad (392)$$

In the following we will express these obstruction in a more familiar way in terms of their characteristic classes. In order to do that, we first need to generalize the discussion to differential  $\mathfrak{g}$ -cocycle. But that is now straightforward.

### 8.2.1 Examples

The continuation of the discussion of 6.3.1 to coset spaces gives a classical illustration of the lifting construction considered here.

**Cohomology of coset spaces.** The above relation between the cohomology of groups and that of their Chevalley-Eilenberg qDGCAs generalizes to coset spaces. This also illustrates the constructions which are discussed later in 8. Consider the case of an ordinary extension of (compact connected) Lie groups:

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1 \quad (393)$$

or even the same sequence in which  $G/H$  is only a homogeneous space and not itself a group. For a closed connected subgroup  $t : H \hookrightarrow G$ , there is the induced map  $Bt : BH \rightarrow BG$  and a commutative diagram

$$\begin{array}{ccc} W(\mathfrak{g}) & \xrightarrow{dt^*} & W(\mathfrak{h}) \\ \uparrow & & \uparrow \\ \wedge^\bullet P_G & \xrightarrow{dt^*} & \wedge^\bullet P_H \end{array} . \quad (394)$$

By analyzing the fibration sequence

$$G/H \rightarrow EG/H \simeq BH \rightarrow BG, \quad (395)$$

Halperin and Thomas [42] show there is a morphism

$$\wedge^\bullet (P_G \oplus Q_H) \rightarrow \Omega^\bullet(G/K) \quad (396)$$

inducing an isomorphism in cohomology. It is not hard to see that their morphism factors through

$$\wedge^\bullet (\mathfrak{g}^* \oplus \mathfrak{h}^*[1]). \quad (397)$$

In general, the homogeneous space  $G/H$  itself is not a group, but in case of an extension  $H \rightarrow G \rightarrow K$ , we also have  $BK$  and the sequences  $K \rightarrow BH \rightarrow BG$  and  $BH \rightarrow BG \rightarrow BK$ . Up to homotopy equivalence, the fiber of the bundle  $BH \rightarrow BG$  is  $K$  and that of  $BG \rightarrow BK$  is  $BH$ . In particular, consider an extension of  $\mathfrak{g}$  by a String-like Lie  $\infty$ -algebra

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{i} \mathrm{CE}(\mathfrak{g}_\mu) \xleftarrow{\quad} \mathrm{CE}(\mathfrak{g}) . \quad (398)$$

Regard  $\mathfrak{g}$  now as the quotient  $\mathfrak{g}_\mu/b^{n-1}\mathfrak{u}(1)$  and recognize that corresponding to  $BH$  we have  $b^n\mathfrak{u}(1)$ . Thus we have a quasi-isomorphism

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1) \hookrightarrow \mathfrak{g}_\mu) \simeq \mathrm{CE}(\mathfrak{g}) \quad (399)$$

and hence a morphism

$$\mathrm{CE}(b^n\mathfrak{u}(1)) \rightarrow \mathrm{CE}(\mathfrak{g}). \quad (400)$$

Given a  $\mathfrak{g}$ -bundle cocycle

$$\begin{array}{ccc} & & \mathrm{CE}(\mathfrak{g}) \\ & \swarrow & \\ & A_{\mathrm{vert}} & \\ \Omega_{\mathrm{vert}}^\bullet(Y) & & \end{array} \quad (401)$$

and given an extension of  $\mathfrak{g}$  by a String-like Lie  $\infty$ -algebra

$$\mathrm{CE}(b^{n-1}\mathfrak{u}(1)) \xleftarrow{i} \mathrm{CE}(\mathfrak{g}_\mu) \xleftarrow{\quad} \mathrm{CE}(\mathfrak{g}) \quad (402)$$

we ask if it is possible to *lift the cocycle* through this extension, i.e. to find a dotted arrow in

$$\begin{array}{ccc} \mathrm{CE}(b^{n-1}\mathfrak{u}(1)) & \xleftarrow{\quad} & \mathrm{CE}(\mathfrak{g}_\mu) \xleftarrow{\quad} \mathrm{CE}(\mathfrak{g}) \\ & \searrow \text{dotted} & \swarrow A_{\mathrm{vert}} \\ & \Omega_{\mathrm{vert}}^\bullet(Y) & \end{array} . \quad (403)$$

In general this is not possible. Indeed, consider the map  $A'_{\mathrm{vert}}$  given by  $\mathrm{CE}(b^n\mathfrak{u}(1)) \rightarrow \mathrm{CE}(\mathfrak{g})$  composed with  $A_{\mathrm{vert}}$ . The nontriviality of the  $b^n\mathfrak{u}(1)$ -cocycle  $A'_{\mathrm{vert}}$  is the obstruction to constructing the desired lift.

### 8.3 Lifts of $\mathfrak{g}$ -connections through String-like extensions

In order to find the obstructing characteristic classes, we would like to extend the above lift 403 of  $\mathfrak{g}$ -descent objects to a lift of  $\mathfrak{g}$ -connection descent objects extending them, according to 7.2. Hence we would like first to *extend*  $A_{\text{vert}}$  to  $(A, F_A)$

$$\begin{array}{ccccc}
 \text{CE}(b^{n-1}\mathfrak{u}(1)) & \longleftarrow & \text{CE}(\mathfrak{g}_\mu) & \longleftarrow & \text{CE}(\mathfrak{g}) \\
 \uparrow & & \uparrow & & \uparrow \\
 W(b^{n-1}\mathfrak{u}(1)) & \longleftarrow & W(\mathfrak{g}_\mu) & \longleftarrow & W(\mathfrak{g}) \\
 \uparrow & & \uparrow & & \uparrow \\
 \text{inv}(b^{n-1}\mathfrak{u}(1)) & \longleftarrow & \text{inv}(\mathfrak{g}_\mu) & \longleftarrow & \text{inv}(\mathfrak{g}) \\
 & & \uparrow & & \uparrow \\
 & & \Omega^\bullet(Y) & & \uparrow \\
 & & \uparrow & & \uparrow \\
 & & \Omega^\bullet(X) & & \uparrow
 \end{array}
 \quad (404)$$

$\swarrow A_{\text{vert}}$  from  $\text{CE}(\mathfrak{g})$  to  $\Omega^\bullet_{\text{vert}}(Y)$   
 $\swarrow (A, F_A)$  from  $W(\mathfrak{g})$  to  $\Omega^\bullet(Y)$   
 $\swarrow \{K_i\}$  from  $\text{inv}(\mathfrak{g})$  to  $\Omega^\bullet(X)$

and then *lift* the resulting  $\mathfrak{g}$ -connection descent object  $(A, F_A)$  to a  $\mathfrak{g}_\mu$ -connection object  $(\hat{A}, F_{\hat{A}})$

$$\begin{array}{ccccc}
 \text{CE}(b^{n-1}\mathfrak{u}(1)) & \longleftarrow & \text{CE}(\mathfrak{g}_\mu) & \longleftarrow & \text{CE}(\mathfrak{g}) \\
 \uparrow & & \uparrow & & \uparrow \\
 W(b^{n-1}\mathfrak{u}(1)) & \longleftarrow & W(\mathfrak{g}_\mu) & \longleftarrow & W(\mathfrak{g}) \\
 \uparrow & & \uparrow & & \uparrow \\
 \text{inv}(b^{n-1}\mathfrak{u}(1)) & \longleftarrow & \text{inv}(\mathfrak{g}_\mu) & \longleftarrow & \text{inv}(\mathfrak{g}) \\
 & & \uparrow & & \uparrow \\
 & & \Omega^\bullet(Y) & & \uparrow \\
 & & \uparrow & & \uparrow \\
 & & \Omega^\bullet(X) & & \uparrow
 \end{array}
 \quad (405)$$

$\swarrow A_{\text{vert}}$  from  $\text{CE}(\mathfrak{g}_\mu)$  to  $\Omega^\bullet_{\text{vert}}(Y)$   
 $\swarrow A_{\text{vert}}$  from  $\text{CE}(\mathfrak{g})$  to  $\Omega^\bullet_{\text{vert}}(Y)$   
 $\swarrow (\hat{A}_{\text{vert}}, F_{\hat{A}_{\text{vert}}})$  from  $W(\mathfrak{g}_\mu)$  to  $\Omega^\bullet(Y)$   
 $\swarrow (A, F_A)$  from  $W(\mathfrak{g})$  to  $\Omega^\bullet(Y)$   
 $\swarrow \{\hat{K}_i\}$  from  $\text{inv}(\mathfrak{g}_\mu)$  to  $\Omega^\bullet(X)$   
 $\swarrow \{K_i\}$  from  $\text{inv}(\mathfrak{g})$  to  $\Omega^\bullet(X)$

The situation is essentially an obstruction problem as before, only that instead of single morphisms, we are now lifting an entire sequence of morphisms. As before, we measure the obstruction to the existence of the

lift by precomposing everything with the a map from a weak cokernel:

The result is a  $b^n \mathfrak{u}(1)$ -connection object. We will call (the class of) this the **generalized Chern-Simons  $(n+1)$ -bundle** obstructing the lift.

In order to construct the lift it is convenient, for similar reasons as in the proof of proposition 23, to work with  $\text{CE}(\text{cs}_P(\mathfrak{g}))$  instead of the isomorphic  $W(\mathfrak{g}_\mu)$ , using the isomorphism from proposition 21. Furthermore, using the identity

$$W(\mathfrak{g}) = \text{CE}(\text{inn}(\mathfrak{g})) \quad (406)$$

mentioned in 6.1, we can hence consider instead of

$$W(b^{n-1}) \longleftarrow W(\mathfrak{g}_\mu) \longleftarrow W(\mathfrak{g}) \quad (407)$$

the sequence

$$\text{CE}(\text{inn}(b^{n-1})) \longleftarrow \text{CE}(\text{cs}_P(\mathfrak{g})) \longleftarrow \text{CE}(\text{inn}(\mathfrak{g})) . \quad (408)$$

Fortunately, this still satisfies the assumptions of proposition 38. So in complete analogy, we find the extension of proposition 41 from  $\mathfrak{g}$ -bundle cocycles to differential  $\mathfrak{g}$ -cocycles:

**Proposition 43** *The morphism*

$$f^{-1} : \text{CE}(\text{inn}(b^{n-1} \mathfrak{u}(1))) \hookrightarrow \text{CE}(\text{cs}_P(\mathfrak{g})) \rightarrow \text{CE}(\text{inn}(\mathfrak{g})) \quad (409)$$

*constructed as in proposition 41 acts as the identity on  $\mathfrak{g}^* \oplus \mathfrak{g}^*[1]$*

$$f^{-1}|_{\mathfrak{g}^* \oplus \mathfrak{g}^*[1]} = \text{Id} \quad (410)$$

*and satisfies*

$$f^{-1}|_{\mathbb{R}[n+2]} : c \mapsto P . \quad (411)$$

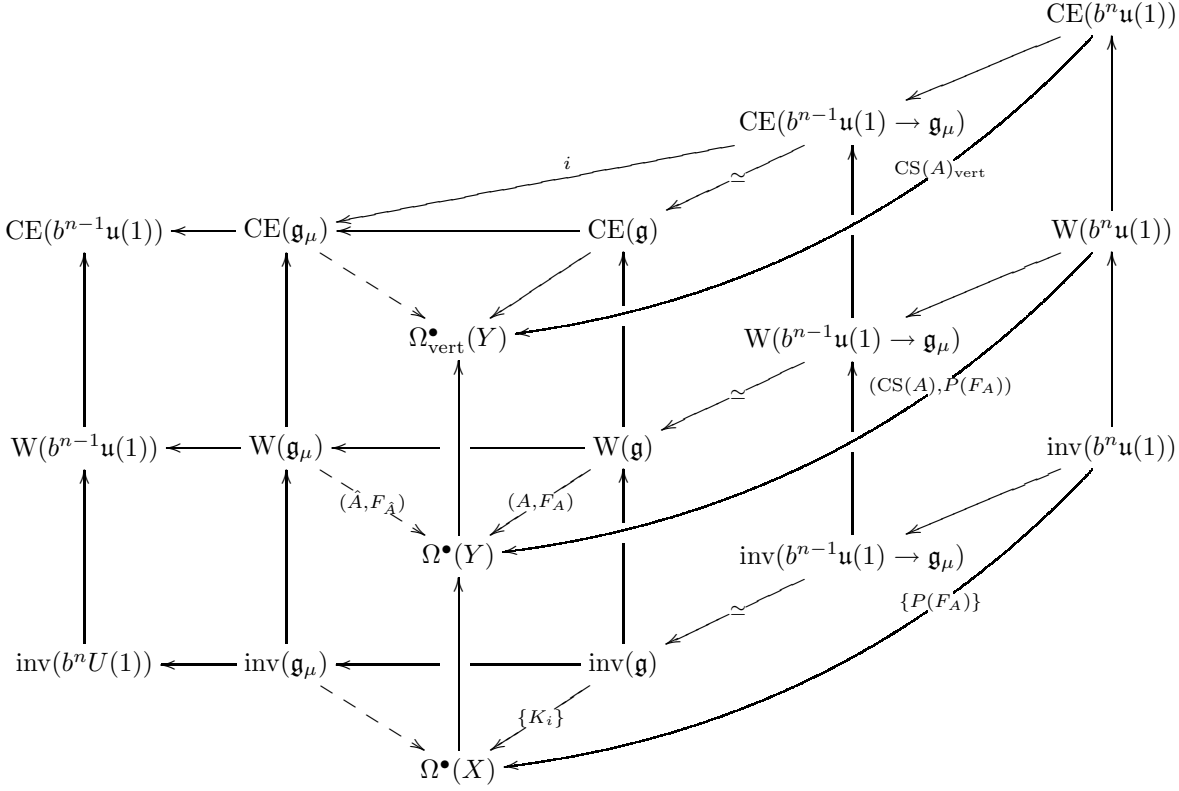


Figure 12: The **generalized Chern-Simons  $b^n \mathfrak{u}(1)$ -bundle that obstructs the lift of a given  $\mathfrak{g}$ -bundle to a  $\mathfrak{g}_\mu$ -bundle**, or rather the descent object representing it.

This means that, as an extension of proposition 42, we find the differential  $b^n \mathfrak{u}(1)$   $(n+1)$ -cocycle

$$\Omega^\bullet(Y) \xleftarrow{\hat{A}} W(b^n \mathfrak{u}(1)) \quad (412)$$

obstructing the lift of a differential  $\mathfrak{g}$ -cocycle

$$\Omega^\bullet(Y) \xleftarrow{(A, F_A)} W(\mathfrak{g}) , \quad (413)$$

according to the above discussion

$$\begin{array}{c} \text{CE}(\text{inn}(b^{n-1} \mathfrak{u}(1)) \hookrightarrow \text{inn}(\mathfrak{g}_\mu)) \xleftarrow{j} W(b^n \mathfrak{u}(1)) , \\ \text{W}(b^{n-1} \mathfrak{u}(1)) \xleftarrow{i^*} \text{W}(\mathfrak{g}_\mu) \xleftarrow{\quad} \text{W}(\mathfrak{g}) \xleftarrow{f^{-1}} \text{W}(b^n \mathfrak{u}(1)) , \\ \text{W}(\mathfrak{g}_\mu) \xleftarrow{(\hat{A}, F_{\hat{A}})} \text{W}(\mathfrak{g}) \xleftarrow{(A, F_A)} \Omega^\bullet(Y) \end{array} \quad (414)$$

to be the connection  $(n+1)$ -form

$$\hat{A} = \text{CS}(A) \in \Omega^{n+1}(Y) \quad (415)$$





where, as discussed above, we are making use of the isomorphism  $W(\mathfrak{g}_\mu) \simeq CE(cs_P(\mathfrak{g}))$  from proposition 21. The crucial aspect of this composite is the isomorphism

$$W(\mathfrak{g}) \xleftarrow[\simeq]{f^{-1}} (W(bu(1)) \rightarrow CE_P(\mathfrak{g})) \quad (423)$$

from proposition 40. This is where the obstruction data is picked up. The important formula governing this is equation 346, which describes how the shifted elements coming from  $W(bu(1))$  in the mapping cone  $(W(bu(1)) \rightarrow CE_P(\mathfrak{g}))$  are mapped to  $W(\mathfrak{g})$ .

Recall that  $W(b^2u(1)) = F(\mathbb{R}[3])$  is generated from elements  $(h, dh)$  of degree 3 and 4, respectively, that  $W(bu(1)) = F(\mathbb{R}[2])$  is generated from elements  $(c, dc)$  of degree 2 and 3, respectively, and that  $CE(cs_P(\mathfrak{g}))$  is generated from  $\mathfrak{g}^* \oplus \mathfrak{g}^*[1]$  together with elements  $b$  and  $c$  of degree 2 and 3, respectively, with

$$d_{CE(cs_P(\mathfrak{g}))}b = c - cs \quad (424)$$

and

$$d_{CE(cs_P(\mathfrak{g}))}c = P, \quad (425)$$

where  $cs \in \wedge^3(\mathfrak{g}^* \oplus \mathfrak{g}^*[1])$  is the transgression element interpolating between the cocycle  $\mu = \langle \cdot, [\cdot, \cdot] \rangle \in \wedge^3(\mathfrak{g}^*)$  and the invariant polynomial  $P = \langle \cdot, \cdot \rangle \in \wedge^2(\mathfrak{g}^*[1])$ . Hence the map  $f^{-1}$  acts as

$$f^{-1} : \sigma b \mapsto -(d_{CE(cs_P(\mathfrak{g}))}b)|_{\wedge \bullet (\mathfrak{g}^* \oplus \mathfrak{g}^*[1])} = +cs \quad (426)$$

and

$$f^{-1} : \sigma c \mapsto -(d_{CE(cs_P(\mathfrak{g}))}c)|_{\wedge \bullet (\mathfrak{g}^* \oplus \mathfrak{g}^*[1])} = -P. \quad (427)$$

Therefore the above composite  $(H_{(3)}, G_{(4)})$  maps the generators  $(h, dh)$  of  $W(b^2u(1))$  as

$$\begin{array}{ccc} & h & \\ & \swarrow \sigma b & \\ & \sigma b & \\ & \swarrow \sim & \\ & cs & \\ & \swarrow (A, F_A) & \\ CS_P(F_A) & \xleftarrow{(H_{(3)}, G_{(4)})} & h \end{array} \quad (428)$$

and

$$\begin{array}{ccc} & dh & \\ & \swarrow -\sigma c & \\ & -\sigma c & \\ & \swarrow \sim & \\ & P & \\ & \swarrow (A, F_A) & \\ P(F_A) & \xleftarrow{(H_{(3)}, G_{(4)})} & dh \end{array} \quad (429)$$

Notice the signs here, as discussed around equation 374. We then have that the connection 3-form of the Chern-Simons 3-bundle given by our obstructing  $b^2u(1)$ -connection descent object is the Chern-Simons form

$$H_{(3)} = -CS(A, F_A) = -\langle A \wedge dA \rangle - \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \in \Omega^3(Y) \quad (430)$$

of the original Ehresmann connection 1-form  $A$ , and its 4-form curvature is therefore the corresponding 4-form

$$G_{(4)} = -P(F_A) = \langle F_A \wedge F_A \rangle \in \Omega^4(Y). \quad (431)$$

This descends down to  $X$ , where it constitutes the characteristic form which classifies the obstruction. Indeed, noticing that  $\text{inv}(b^2\mathbf{u}(1)) = \wedge^\bullet(\mathbb{R}[4])$ , we see that (this works the same for all line  $n$ -bundles, i.e., for all  $b^{n-1}\mathbf{u}(1)$ -connection descent objects) the characteristic forms of the obstructing Chern-Simons 3-bundle

$$\begin{array}{c}
 \text{inv}(b^2\mathbf{u}(1)) \\
 \swarrow \text{inv}(bu(1) \rightarrow \mathfrak{g}_\mu) \\
 \swarrow \simeq \\
 \text{inv}(\mathfrak{g}) \\
 \swarrow \{K_i\} \\
 \Omega^\bullet(X)
 \end{array}
 \quad
 \begin{array}{c}
 \text{inv}(b^2\mathbf{u}(1)) \\
 \searrow \\
 \{G_{(4)}\} \\
 \swarrow \\
 \Omega^\bullet(X)
 \end{array}
 \quad (432)$$

consist only and precisely of this curvature 4-form: the second Chern-form of the original  $G$ -bundle  $P$ .

## 9 $L_\infty$ -algebra parallel transport

One of the main points about a connection is that it allows to do parallel transport. Connections on ordinary bundles give rise to a notion of parallel transport along curves, known as holonomy if these curves are closed.

Higher connections on  $n$ -bundles should yield a way to obtain a notion of parallel transport over  $n$ -dimensional spaces. In physics, this assignment plays the role of the gauge coupling term in the non-kinetic part of the action functional: the action functional of the charged particle is essentially its parallel transport with respect to an ordinary (1-)connection, while the action functional of the string contains the parallel transport of a 2-connection (the Kalb-Ramond field). Similarly the action functional of the membrane contains the parallel transport of a 3-connection (the supergravity “ $C$ -field”).

There should therefore be a way to assign to any one of our  $\mathfrak{g}$ -connection descent objects for  $\mathfrak{g}$  any Lie  $n$ -algebra

- a prescription for parallel transport over  $n$ -dimensional spaces;
- a configuration space for the  $n$ -particle coupled to that transport;
- a way to transgress the transport to an action functional on that configuration space;
- a way to obtain the corresponding quantum theory.

Each point separately deserves a separate discussion, but in the remainder we shall quickly give an impression for how each of these points is addressed in our context.

### 9.1 $L_\infty$ -parallel transport

In this section we indicate briefly how our notion of  $\mathfrak{g}$ -connections give rise to a notion of parallel transport over  $n$ -dimensional spaces. The abelian case (meaning here that  $\mathfrak{g}$  is an  $L_\infty$  algebra such that  $\text{CE}(\mathfrak{g})$  has trivial differential) is comparatively easy to discuss. It is in fact the only case considered in most of the literature. Nonabelian parallel  $n$ -transport in the integrated picture for  $n$  up to 2 is discussed in [9, 68, 69, 70]. There is a close relation between all differential concepts we develop here and the corresponding integrated concepts, but here we will not attempt to give a comprehensive discussion of the translation.

Given an  $(n-1)$ -brane (“ $n$ -particle”) whose  $n$ -dimensional worldvolume is modeled on the smooth parameter space  $\Sigma$  (for instance  $\Sigma = T^2$  for the closed string) and which propagates on a target space  $X$  in that its configurations are given by maps

$$\phi : \Sigma \rightarrow X \quad (433)$$

hence by dg-algebra morphisms

$$\Omega^\bullet(\Sigma) \xleftarrow{\phi^*} \Omega^\bullet(X) \quad (434)$$

we can couple it to a  $\mathfrak{g}$ -descent connection object  $(Y, (A, F_A))$  over  $X$  pulled back to  $\Sigma$  if  $Y$  is such that for every map

$$\phi : \Sigma \rightarrow X \quad (435)$$

the pulled back surjective submersion has a global section

$$\begin{array}{ccc} & \phi^* Y & \\ \hat{\phi} \nearrow & \downarrow \pi & \\ \Sigma & \xrightarrow{\text{Id}} & \Sigma \end{array} . \quad (436)$$

**Definition 40 (parallel transport)** *Given a  $\mathfrak{g}$ -descent object  $(Y, (A, F_A))$  on a target space  $X$  and a parameter space  $\Sigma$  such that for all maps  $\phi : \Sigma \rightarrow X$  the pullback  $\phi^* Y$  has a global section, we obtain a map*

$$\text{tra}_{(A)} : \text{Hom}_{\text{DGCA}}(\Omega^\bullet(X), \Omega^\bullet(\Sigma)) \rightarrow \text{Hom}_{\text{DGCA}}(W(\mathfrak{g}), \Omega^\bullet(\Sigma)) \quad (437)$$

by precomposition with

$$\Omega^\bullet(Y) \xleftarrow{(A, F_A)} W(\mathfrak{g}) . \quad (438)$$

This is essentially the parallel transport of the  $\mathfrak{g}$ -connection object  $(Y, (A, F_A))$ . A full discussion is beyond the scope of this article, but for the special case that our  $L_\infty$ -algebra is  $(n-1)$ -fold shifted  $\mathfrak{u}(1)$ ,  $\mathfrak{g} = b^{n-1}\mathfrak{u}(1)$ , the elements in

$$\text{Hom}_{\text{dgca}}(W(\mathfrak{g}), \Omega^\bullet(\Sigma)) = \Omega^\bullet(\Sigma, b^{n-1}\mathfrak{u}(1)) \simeq \Omega^n(\Sigma) \quad (439)$$

are in bijection with  $n$ -forms on  $\Sigma$ . Therefore they can be integrated over  $\Sigma$ . Then the functional

$$\int_\Sigma \text{tra}_A : \text{Hom}_{\text{dgca}}(\Omega^\bullet(Y), \Omega^\bullet(\Sigma)) \rightarrow \mathbb{R} \quad (440)$$

is the full parallel transport of  $A$ .

**Proposition 45** *The map  $\text{tra}_{(A)}$  is indeed well defined, in that it depends at most on the homotopy class of the choice of global section  $\hat{\phi}$  of  $\phi$ .*

Proof. Let  $\hat{\phi}_1$  and  $\hat{\phi}_2$  be two global sections of  $\phi^* Y$ . Let  $\hat{\phi} : \Sigma \times I \rightarrow \phi^* Y$  be a homotopy between them, i.e. such that  $\hat{\phi}|_0 = \hat{\phi}_1$  and  $\hat{\phi}|_1 = \hat{\phi}_2$ . Then the difference in the parallel transport using  $\hat{\phi}_1$  and  $\hat{\phi}_2$  is the integral of the pullback of the curvature form of the  $\mathfrak{g}$ -descent object over  $\Sigma \times I$ . But that vanishes, due to the commutativity of

$$\begin{array}{ccccc} & \Omega^\bullet(\phi^* Y) & \xleftarrow{(A, F_A)} & W(\mathfrak{g}) & \\ & \uparrow \hat{\phi}^* & & \uparrow & \\ \Omega^\bullet(\Sigma \times I) & \xleftarrow{\phi^*} & \Omega^\bullet(\Sigma) & \xleftarrow{K} & \text{inv}(b^{n-1}\mathfrak{u}(1)) = b^n \mathfrak{u}(1) \\ & & \searrow & \nearrow & \\ & & 0 & & \end{array} \quad (441)$$

The composite of the morphisms on the top boundary of this diagram send the single degree  $(n+1)$ -generator of  $\text{inv}(b^{n-1}\mathbf{u}(1)) = \text{CE}(b^n\mathbf{u}(1))$  to the curvature form of the  $\mathfrak{g}$ -connection descent object pulled back to  $\Sigma$ . It is equal to the composite of the horizontal morphisms along the bottom boundary by the definition of  $\mathfrak{g}$ -descent objects. These vanish, as there is no nontrivial  $(n+1)$ -form on the  $n$ -dimensional  $\Sigma$ .  $\square$

### 9.1.1 Examples.

#### Chern-Simons and higher Chern-Simons action functionals

**Proposition 46** *For  $G$  simply connected, the parallel transport coming from the Chern-Simons 3-bundle discussed in 8.3.1 for  $\mathfrak{g} = \text{Lie}(G)$  reproduces the familiar Chern-Simons action functional [31]*

$$\int_{\Sigma} \left( \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \right) \quad (442)$$

over 3-dimensional  $\Sigma$ .

Proof. Recall from 8.3.1 that we can build the connection descent object for the Chern-Simons connection on the surjective submersion  $Y$  coming from the total space  $P$  of the underlying  $G$ -bundle  $P \rightarrow X$ . Then  $\phi^*Y = \phi^*P$  is simply the pullback of that  $G$ -bundle to  $\Sigma$ . For  $G$  simply connected,  $BG$  is 3-connected and hence any  $G$ -bundle on  $\Sigma$  is trivializable. Therefore the required lift  $\hat{\phi}$  exists and we can construct the above diagram. By equation 430 one sees that the integral which gives the parallel transport is indeed precisely the Chern-Simons action functional.  $\square$

Higher Chern-Simons  $n$ -bundles, coming from obstructions to fivebrane lifts or still higher lifts, similarly induce higher dimensional generalizations of the Chern-Simons action functional.

**BF-theoretic functionals** From proposition 19 it follows that we can similarly obtain the action functional of BF theory, discussed in 6.6, as the parallel transport of the 4-connection descent object which arises as the obstruction to lifting a 2-connection descent object for a strict Lie 2-algebra  $(\mathfrak{h} \xrightarrow{t} \mathfrak{g})$  through the string-like extension

$$b^2\mathbf{u}(1) \rightarrow (\mathfrak{h} \xrightarrow{t} \mathfrak{g})_{d_{\text{CE}(\mathfrak{h} \xrightarrow{t} \mathfrak{g})}} \mu \rightarrow (\mathfrak{h} \xrightarrow{t} \mathfrak{g}) \quad (443)$$

for  $\mu$  the 3-cocycle on  $\mu$  which transgresses to the invariant polynomial  $P$  on  $\mathfrak{g}$  which appears in the BF-action functional.

## 9.2 Transgression of $L_{\infty}$ -transport

An important operation on parallel transport is its *transgression* to mapping spaces. This is familiar from simple examples, where for instance  $n$ -forms on some space transgress to  $(n-1)$ -forms on the corresponding loop space. We should think of the  $n$ -form here as a  $b^{n-1}\mathbf{u}(1)$ -connection which transgresses to an  $b^{n-2}\mathbf{u}(1)$  connection on loop space.

This modification of the structure  $L_{\infty}$ -algebra under transgression is crucial. In [69] it is shown that for parallel transport  $n$ -functors ( $n=2$  there), the operation of transgression is a very natural one, corresponding to acting on the transport functor with an inner hom operation. As shown there, this operation automatically induces the familiar pull-back followed by a fiber integration on the corresponding differential form data, and also automatically takes care of the modification of the structure Lie  $n$ -group.

The analogous construction in the differential world of  $L_{\infty}$  algebras we state now, without here going into details about its close relation to [69].

**Definition 41 (transgression of  $\mathfrak{g}$ -connections)** *For any  $\mathfrak{g}$ -connection descent object*

$$\begin{array}{ccc}
 F & \xleftarrow{A_{\text{vert}}} & \text{CE}(\mathfrak{g}) \\
 \uparrow i^* & & \uparrow \\
 P & \xleftarrow{(A, F_A)} & W(\mathfrak{g}) \\
 \uparrow \pi^* & & \uparrow \\
 P_{\text{basic}} & \xleftarrow{\{K_i\}} & \text{inv}(\mathfrak{g})
 \end{array} \tag{444}$$

and any smooth space  $\text{par}$ , we can form the image of the above diagram under the functor

$$\text{maps}(-, \Omega^\bullet(\text{par})) : \text{DGCA}s \rightarrow \text{DGCA}s \tag{445}$$

from definition 4 to obtain the generalized  $\mathfrak{g}$ -connection descent object (according to definition 36)

$$\begin{array}{ccc}
 \text{maps}(F, \Omega^\bullet(\text{par})) & \xleftarrow{\text{tg}_{\text{par}}(A_{\text{vert}})} & \text{maps}(\text{CE}(\mathfrak{g}), \Omega^\bullet(\text{par})) \\
 \uparrow \text{tg}_{\text{par}} i^* & & \uparrow \\
 \text{maps}(P, \Omega^\bullet(\text{par})) & \xleftarrow{\text{tg}_{\text{par}}(A, F_A)} & \text{maps}(W(\mathfrak{g}), \Omega^\bullet(\text{par})) \\
 \uparrow \text{tg}_{\text{par}}(\pi^*) & & \uparrow \\
 \text{maps}(P_{\text{basic}}, \Omega^\bullet(\text{par})) & \xleftarrow{\text{tg}_{\text{par}}(\{K_i\})} & \text{maps}(\text{inv}(\mathfrak{g}), \Omega^\bullet(\text{par}))
 \end{array} \tag{446}$$

This new  $\text{maps}(\text{CE}(\mathfrak{g}), \Omega^\bullet(\text{par}))$ -connection descent object we call the *transgression of the original one to  $\text{par}$* .

The operation of transgression is closely related to that of integration.

### 9.2.1 Examples

**Transgression of  $b^{n-1}\mathfrak{u}(1)$ -connections.** Let  $\mathfrak{g}$  be an  $L_\infty$ -algebra of the form shifted  $\mathfrak{u}(1)$ ,  $\mathfrak{g} = b^{n-1}\mathfrak{u}(1)$ . By proposition 17 the Weil algebra  $W(b^{n-1}\mathfrak{u}(1))$  is the free DGCA on a single degree  $n$ -generator  $b$  with differential  $c := db$ . Recall from 6.5.1 that a DGCA morphism  $W(b^{n-1}\mathfrak{u}(1)) \rightarrow \Omega^\bullet(Y)$  is just an  $n$ -form on  $Y$ . For every point  $y \in \text{par}$  and for every multivector  $v \in \wedge^n T_y \text{par}$  we get a 0-form on the smooth space

$$\text{maps}(W(b^{n-1}\mathfrak{u}(1)), \Omega^\bullet(\text{par})) \tag{447}$$

of all  $n$ -forms on  $\text{par}$ , which we denote

$$A(v) \in \Omega(\text{maps}(W(b^{n-1}\mathfrak{u}(1)), \Omega^\bullet(\text{par}))). \tag{448}$$

This is the 0-form on this space of maps obtained from the element  $b \in W(b^{n-1}\mathfrak{u}(1))$  and the current  $\delta_y$  (the ordinary delta-distribution on 0-forms) according to proposition 1. Its value on any any  $n$ -form  $\omega$  is the value of that form evaluated on  $v$ .

Since this, and its generalizations which we discuss in 9.3.1, is crucial for making contact with standard constructions in physics, it may be worthwhile to repeat that statement more explicitly in terms of components: Assume that  $\text{par} = \mathbb{R}^k$  and for any point  $y$  let  $v$  be the unit in  $\wedge T^n \mathbb{R}^n \simeq \mathbb{R}$ . Then  $A(v)$  is the 0-form on the space of forms which sends any form  $\omega = \omega_{\mu_1 \mu_2 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$  to its component

$$A(v) : \omega \mapsto \omega(y)_{12, \dots, n} . \quad (449)$$

This implies that when a  $b^{n-1}\mathfrak{u}(1)$ -connection is transgressed to the space of maps from an  $n$ -dimensional parameter space  $\text{par}$ , it becomes a map that pulls back functions on the space of  $n$ -forms on  $\text{par}$  to the space of functions on maps from parameter space into target space. But such pullbacks correspond to *functions* (0-forms) on the space of maps  $\text{par} \rightarrow \text{tar}$  with values in the space of  $n$ -forms on  $\text{tra}$ .

### 9.3 Configuration spaces of $L_\infty$ -transport

With the notion of  $\mathfrak{g}$ -connections and their parallel transport and transgression in hand, we can say what it means to *couple an  $n$ -particle/ $(n-1)$ -brane to a  $\mathfrak{g}$ -connection*.

**Definition 42 (the charged  $n$ -particle/ $(n-1)$ -brane)** *We say a charged  $n$ -particle/ $(n-1)$ -brane is a tuple  $(\text{par}, (A, F_A))$  consisting of*

- **parameter space**  $\text{par}$ : *a smooth space*
- **a background field**  $(A, F_A)$ : *a  $\mathfrak{g}$ -connection descent object involving*
  - **target space**  $\text{tar}$ : *the smooth space that the  $\mathfrak{g}$ -connection  $(A, F_A)$  lives over;*
  - **space of phases**  $\text{phas}$ : *the smooth space such that  $\Omega^\bullet(\text{phas}) \simeq \text{CE}(\mathfrak{g})$*

*From such a tuple we form*

- **configuration space**  $\text{conf} = \text{hom}_{S^\infty}(\text{par}, \text{tar})$ ;
- **the action functional**  $\exp(S) := \text{tg}_{\text{par}}$ : *the transgression of the background field to configuration space.*

The configuration space thus defined automatically comes equipped with a notion of vertical derivations as described in 5.3.

$$\begin{array}{ccc}
 & 0 & \\
 & \curvearrowright & \\
 \text{maps}(F, \Omega^\bullet(\text{par})) & \Downarrow_{\rho'} & \text{maps}(F, \Omega^\bullet(\text{par})) \\
 & \curvearrowleft & \\
 & [d, \rho'] & \\
 & 0 & \\
 & \curvearrowright & \\
 \text{maps}(P, \Omega^\bullet(\text{par})) & \Downarrow_{\rho} & \text{maps}(P, \Omega^\bullet(\text{par})) \\
 & \curvearrowleft & \\
 & [d, \rho] & 
 \end{array} . \quad (450)$$

These form

- **the gauge symmetries**  $\mathfrak{g}_{\text{gauge}}$ : *an  $L_\infty$ -algebra.*

These act on the horizontal elements of configuration space, which form

- the **anti-fields and anti-ghosts**

in the language of BRST-BV-quantization [74].

We will not go into further details of this here, except for spelling out, as the archetypical example, some details of the computation of the configuration space of ordinary gauge theory.

### 9.3.1 Examples

**Configuration space of ordinary gauge theory.** We compute here the the configuration space of ordinary gauge theory on a manifold  $\text{par}$  with respect to an ordinary Lie algebra  $\mathfrak{g}$ . A configuration of such a theory is a  $\mathfrak{g}$ -valued differential form on  $\text{par}$ , hence, according to 6.5, an element in  $\text{Hom}_{\text{DGCAs}}(W(\mathfrak{g}), \Omega^\bullet(\text{par}))$ . So we are interested in understanding the smooth space

$$\text{maps}(W(\mathfrak{g}), \Omega^\bullet(\text{par})) =: \Omega^\bullet(\text{par}, \mathfrak{g}) \quad (451)$$

according to definition 4, and the differential graded-commutative algebra

$$\text{maps}(W(\mathfrak{g}), \Omega^\bullet(\text{par})) =: \Omega^\bullet(\Omega^\bullet(\text{par}, \mathfrak{g})) \quad (452)$$

of differential forms on it.

To make contact with the physics literature, we describe everything in components. So let  $\text{par} = \mathbb{R}^n$  and let  $\{x^\mu\}$  be the canonical set of coordinate functions on  $\text{par}$ . Choose a basis  $\{t_a\}$  of  $\mathfrak{g}$  and let  $\{t^a\}$  be the corresponding dual basis of  $\mathfrak{g}^*$ . Denote by

$$\delta_y \iota_{\frac{\partial}{\partial x^\mu}} \quad (453)$$

the delta-current on  $\Omega^\bullet(\text{par})$ , according to definition 5, which sends a 1-form  $\omega$  to

$$\omega_\mu(y) := \omega\left(\frac{\partial}{\partial x^\mu}\right)(y). \quad (454)$$

**Summary of the structure of forms on configuration space of ordinary gauge theory.** Recall that the Weil algebra  $W(\mathfrak{g})$  is generated from the  $\{t^a\}$  in degree 1 and the  $\sigma t^a$  in degree 2, with the differential defined by

$$dt^a = -\frac{1}{2} C^a_{bc} t^b \wedge t^c + \sigma t^a \quad (455)$$

$$d(\sigma t^a) = -C^a_{bc} t^b \wedge (\sigma t^c). \quad (456)$$

We will find that  $\text{maps}(W(\mathfrak{g}), \Omega^\bullet(\text{par}))$  does look pretty much entirely like this, only that all generators are now forms on  $\text{par}$ . See table 6.

<b>fields</b>	$\{A_\mu^a(y), (F_A)_{\mu\nu}(y) \in \Omega^0(\Omega(\text{par}, \mathfrak{g})) \mid y \in \text{par}, \mu, \nu \in \{1, \dots, \dim(\text{par}), a \in \{1, \dots, \dim(\mathfrak{g})\}\}\}$
<b>ghosts</b>	$\{c^a(y) \in \Omega^1(\Omega(\text{par}, \mathfrak{g})) \mid y \in \text{par}, a \in \{1, \dots, \dim(\mathfrak{g})\}\}$
<b>antifields</b>	$\{\iota_{(\delta A_\mu^a(y))} \in \text{Hom}(\Omega^1(\Omega(\text{par}, \mathfrak{g})), \mathbb{R}) \mid y \in \text{par}, \mu \in \{1, \dots, \dim(\text{par}), a \in \{1, \dots, \dim(\mathfrak{g})\}\}\}$
<b>anti-ghosts</b>	$\{\iota_{(\beta^a(y))} \in \text{Hom}(\Omega^2(\Omega(\text{par}, \mathfrak{g})), \mathbb{R}) \mid y \in \text{par}, \dim(\text{par}), a \in \{1, \dots, \dim(\mathfrak{g})\}\}$

Table 6: **The BRST-BV field content of gauge theory** obtained from our almost internal hom of dg-algebras, definition 4. The dg-algebra  $\text{maps}(W(\mathfrak{g}), \Omega^\bullet(\text{par}))$  is the algebra of differential forms on a smooth space of maps from  $\text{par}$  to the smooth space underlying  $W(\mathfrak{g})$ . In the above table  $\beta$  is a certain 2-form that one finds in this algebra of forms on the space of  $\mathfrak{g}$ -valued forms.

**Remark.** Before looking at the details of the computation, recall from 5.1 that an  $n$ -form  $\omega$  in  $\text{maps}(\mathbf{W}(\mathfrak{g}), \Omega^\bullet(\text{par}))$  is an assignment

$$\begin{array}{ccccc}
 U & & \text{Hom}_{\text{DGCA}s}(\mathbf{W}(\mathfrak{g}), \Omega^\bullet(\text{par} \times U)) & \xrightarrow{\omega_U} & \Omega^\bullet(U) \\
 \downarrow \phi & & \uparrow \phi^* & & \uparrow \phi^* \\
 V & & \text{Hom}_{\text{DGCA}s}(\mathbf{W}(\mathfrak{g}), \Omega^\bullet(\text{par} \times V)) & \xrightarrow{\omega_V} & \Omega^\bullet(V)
 \end{array} \tag{457}$$

of forms on  $U$  to  $\mathfrak{g}$ -valued forms on  $\text{par} \times U$  for all plot domains  $U$  (subsets of  $\mathbb{R} \cup \mathbb{R}^2 \cup \dots$  for us), natural in  $U$ . We concentrate on those  $n$ -forms  $\omega$  which arise in the way of proposition 1.

**0-Forms.** The 0-forms on the space of  $\mathfrak{g}$ -value forms are constructed as in proposition 1 from an element  $t^a \in \mathfrak{g}^*$  and a current  $\delta_y \iota_{\frac{\partial}{\partial x^\mu}}$  using

$$t^a \delta_y \iota_{\frac{\partial}{\partial x^\mu}} \tag{458}$$

and from an element  $\sigma t^a \in \mathfrak{g}^*[1]$  and a current

$$\delta_y \iota_{\frac{\partial}{\partial x^\mu}} \iota_{\frac{\partial}{\partial x^\nu}} . \tag{459}$$

This way we obtain the families of functions (0-forms) on the space of  $\mathfrak{g}$ -valued forms:

$$A_\mu^a(y) : (\Omega^\bullet(\text{par} \times U) \leftarrow \mathbf{W}(\mathfrak{g}) : A) \mapsto (u \mapsto \iota_{\frac{\partial}{\partial x^\mu}} A(t^a)(y, u)) \tag{460}$$

and

$$F_{\mu\nu}^a(y) : (\Omega^\bullet(\text{par} \times U) \leftarrow \mathbf{W}(\mathfrak{g}) : F_A) \mapsto (u \mapsto \iota_{\frac{\partial}{\partial x^\mu}} \iota_{\frac{\partial}{\partial x^\nu}} F_A(\sigma t^a)(y, u)) \tag{461}$$

which pick out the corresponding components of the  $\mathfrak{g}$ -valued 1-form and of its curvature 2-form, respectively. These are the *fields* of ordinary gauge theory.

**1-Forms.** A 1-form on the space of  $\mathfrak{g}$ -valued forms is obtained from either starting with a degree 1 element and contracting with a degree 0 delta-current

$$t^a \delta_y \tag{462}$$

or starting with a degree 2 element and contracting with a degree 1 delta current:

$$(\sigma t^a) \delta_y \frac{\partial}{\partial x^\mu} . \tag{463}$$

To get started, consider first the case where  $U = I$  is the interval. Then a DGCA morphism

$$(A, F_A) : \mathbf{W}(\mathfrak{g}) \rightarrow \Omega^\bullet(\text{par}) \otimes \Omega^\bullet(I) \tag{464}$$

can be split into its components proportional to  $dt \in \Omega^\bullet(I)$  and those not containing  $dt$ . We can hence write the general  $\mathfrak{g}$ -valued 1-form on  $\text{par} \times I$  as

$$(A, F_A) : t^a \mapsto A^a(y, t) + g^a(y, t) \wedge dt \tag{465}$$

and the corresponding curvature 2-form as

$$(A, F_A) : \sigma t^a \mapsto (d_{\text{par}} + d_t)(A^a(y, t) + g^a(y, t) \wedge dt) + \frac{1}{2} C_{bc}^a (A^a(y, t) + g^a(y, t) \wedge dt) \wedge (A^b(y, t) + g^b(y, t) \wedge dt)$$



$$= F_A^a(y, t) + (\partial_t A^a(y, t) + d_{\text{par}} g^a(y, t) + [g, A]^a) \wedge dt. \quad (466)$$

By contracting this again with the current  $\delta_y \frac{\partial}{\partial x^\mu}$  we obtain the 1-forms

$$t \mapsto g^a(y, t) dt \quad (467)$$

and

$$t \mapsto (\partial_t A_\mu^a(y, t) + \partial_\mu g^a(y, t) + [g, A_\mu]^a) dt \quad (468)$$

on the interval. We will identify the first one with the component of the 1-forms on the space of  $\mathfrak{g}$ -valued forms on par called the *ghosts* and the second one with the 1-forms which are killed by the objects called the *anti-fields*.

To see more of this structure, consider now  $U = I^2$ , the unit square. Then a DGCA morphism

$$(A, F_A) : W(\mathfrak{g}) \rightarrow \Omega^\bullet(\text{par}) \otimes \Omega^\bullet(I^2) \quad (469)$$

can be split into its components proportional to  $dt^1, dt^2 \in \Omega^\bullet(I^2)$ . We hence can write the general  $\mathfrak{g}$ -valued 1-form on  $Y \times I$  as

$$(A, F_A) : t^a \mapsto A^a(y, t) + g_i^a(y, t) \wedge dt^i, \quad (470)$$

and the corresponding curvature 2-form as

$$\begin{aligned} (A, F_A) : \sigma t^a &\mapsto (d_Y + d_{I^2})(A^a(y, t) + g_i^a(y, t) \wedge dt^i) \\ &+ \frac{1}{2} C_{bc}^a (A^a(y, t) + g_i^a(y, t) \wedge dt^i) \wedge (A^b(y, t) + g_j^b(y, t) \wedge dt^j) \\ &= F_A^a(y, t) + (\partial_{t^i} A^a(y, t) + d_Y g_i^a(y, t) + [g_i, A]^a) \wedge dt^i \\ &\quad + (\partial_i g_j^a + [g_i, g_j]^a) dt^i \wedge dt^j. \end{aligned} \quad (471)$$

By contracting this again with the current  $\delta_y \frac{\partial}{\partial x^\mu}$  we obtain the 1-forms

$$t \mapsto g_i^a(y, t) dt^i \quad (472)$$

and

$$t \mapsto (\partial_t A_\mu^a(y, t) + \partial_\mu g_i^a(y, t) + [g_i, A_\mu]^a) dt^i \quad (473)$$

on the unit square. These are again the local values of our

$$c^a(y) \in \Omega^1(\Omega^\bullet(\text{par}, \mathfrak{g})) \quad (474)$$

and

$$\delta A_\mu^a(Y) \in \Omega^1(\Omega^\bullet(\text{par}, \mathfrak{g})). \quad (475)$$

The second 1-form vanishes in directions in which the variation of the  $\mathfrak{g}$ -valued 1-form  $A$  is a pure gauge transformation induced by the function  $g^a$  which is measured by the first 1-form. Notice that it is the sum of the exterior derivative of the 0-form  $A_\mu^a(y)$  with another term.

$$\delta A_\mu^a(y) = d(A_\mu^a(y)) + \delta_g A_\mu^a(y). \quad (476)$$

The first term on the right measures the change of the connection, the second subtracts the contribution to this change due to gauge transformations. So the 1-form  $\delta A_\mu^a(y)$  on the space of  $\mathfrak{g}$ -valued forms vanishes along all directions along which the form  $A$  is modified purely by a gauge transformation. The  $\delta A_\mu^a(y)$  are the 1-forms the pairings dual to which will be the *antifields*.

**2-Forms.** We have already seen the 2-form appear on the standard square. We call this 2-form

$$\beta^a \in \Omega^2(\Omega^\bullet(\text{par}, \mathfrak{g})), \quad (477)$$

corresponding on the unit square to the assignment

$$\beta^a : (\Omega^\bullet(\text{par} \times I^2) \leftarrow W(\mathfrak{g}) : A) \mapsto (\partial_i g_j^a + [g_i, g_j]^a) dt^i \wedge dt^j. \quad (478)$$

There is also a 2-form coming from  $(\sigma t^a) \delta_y$ . Then one immediately sees that our forms on the space of  $\mathfrak{g}$ -valued forms satisfy the relations

$$dc^a(y) = -\frac{1}{2} C^a_{bc} c^b(y) \wedge c^c(y) + \beta^a(y) \quad (479)$$

$$d\beta^a(y) = -C^a_{bc} c^b(y) \wedge \beta^c(y). \quad (480)$$

The 2-form  $\beta$  on the space of  $\mathfrak{g}$ -valued forms is what is being contracted by the horizontal pairings called the *antighosts*. We see, in total, that  $\Omega^\bullet(\Omega^\bullet(\text{par}, \mathfrak{g}))$  is the Weil algebra of a DGCA, which is obtained from the above formulas by setting  $\beta = 0$  and  $\delta A = 0$ . This DGCA is the algebra of the gauge groupoid, that where the only morphisms present are gauge transformations.

The computation we have just performed are over  $U = I^2$ . However, it should be clear how this extends to the general case.

**Chern-Simons theory.** One can distinguish two ways to set up Chern-Simons theory. In one approach one regards principal  $G$ -bundles on abstract 3-manifolds, in the other approach one fixes a given principal  $G$ -bundle  $P \rightarrow X$  on some base space  $X$ , and pulls it back to 3-manifolds equipped with a map into  $X$ . Physically, the former case is thought of as Chern-Simons theory proper, while the latter case arises as the gauge coupling part of the membrane propagating on  $X$ . One tends to want to regard the first case as a special case of the second, obtained by letting  $X = BG$  be the classifying space for  $G$ -bundles and  $P$  the universal  $G$ -bundle on that.

In our context this is realized by proposition 34, which gives the canonical Chern-Simons 3-bundle on  $BG$  in terms of a  $b^2\mathfrak{u}(1)$ -connection descent object on  $W(\mathfrak{g})$ . Picking some 3-dimensional parameter space manifold  $\text{par}$ , we can transgress this  $b^2\mathfrak{u}(1)$ -connection to the configuration space  $\text{maps}(W(\mathfrak{g}), \Omega^\bullet(\text{par}))$ , which we learned is the configuration space of ordinary gauge theory.

$$\begin{array}{ccc} \text{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \text{CE}(b^{n-1}\mathfrak{u}(1)) \\ \uparrow i^* & & \uparrow \\ W(\mathfrak{g}) & \xleftarrow{(cs, P)} & W(b^{n-1}\mathfrak{u}(1)) \\ \uparrow & & \uparrow \\ \text{inv}(\mathfrak{g}) & \xleftarrow{P} & \text{inv}(b^n\mathfrak{u}(1)) \end{array} \quad \mapsto \quad \begin{array}{ccc} \text{maps}(\text{CE}(\mathfrak{g}), \Omega^\bullet(\text{par})) & \xleftarrow{\text{tg}_{\text{par}} \mu} & \text{maps}(\text{CE}(b^{n-1}\mathfrak{u}(1)), \Omega^\bullet(\text{par})) \\ \uparrow i^* & & \uparrow \\ \text{maps}(W(\mathfrak{g}), \Omega^\bullet(\text{par})) & \xleftarrow{\text{tg}_{\text{par}} (cs, P)} & \text{maps}(W(b^{n-1}\mathfrak{u}(1)), \Omega^\bullet(\text{par})) \\ \uparrow & & \uparrow \\ \text{maps}(\text{inv}(\mathfrak{g}), \Omega^\bullet(\text{par})) & \xleftarrow{\text{tg}_{\text{par}} P} & \text{maps}(\text{CE}(b^{n-1}\mathfrak{u}(1)), \Omega^\bullet(\text{par})) \end{array}.$$

Proposition 46 says that the transgressed connection is the Chern-Simons action functional.

Further details of this should be discussed elsewhere.

**Transgression of  $p$ -brane structures to loop space** It is well known that obstructions to String structures on a space  $X$  – for us: Chern-Simons 3-bundles as in 8 – can be conceived

- either in terms of a 3-bundle on  $X$  classified by a four class on  $X$  obstructing the lift of a 1-bundle on  $X$  to a 2-bundle;
- or in terms of a 2-bundle on  $LX$  classified by a 3-class on  $LX$  obstructing the lift of a 1-bundle on  $LX$  to another 1-bundle, principal for a Kac-Moody central extension of the loop group.

In the second case, one is dealing with the transgression of the first case to loop space.

The relation between the two points of views is carefully described in [56]. Essentially, the result is that *rationally* both obstructions are equivalent.

**Remark.** Unfortunately, there is no universal agreement on the convention of the direction of the operation called transgression. Both possible conventions are used in the literature relevant for our purpose here. For instance [18] say transgression for what [2] calls the inverse of transgression (which, in turn, should be called suspension).

We will demonstrate in the context of  $L_\infty$ -algebra connections how Lie algebra  $(n+1)$ -cocycles related to  $p$ -brane structures on  $X$  transgress to loop Lie algebra  $n$ -cocycles on loop space. One can understand this also as an alternative proof of the strictification theorem of the String Lie 2-algebra (proposition 25), but this will not be further discussed here.

So let  $\mathfrak{g}$  be an ordinary Lie algebra,  $\mu$  an  $(n+1)$ -cocycle on it in transgression with an invariant polynomial  $P$ , where the transgression is mediated by the transgression element  $cs$  as described in 6.3.

According to proposition 34 the corresponding universal obstruction structure is the  $b^n \mathfrak{u}(1)$ -connection

$$\begin{array}{ccc}
 \text{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \text{CE}(b^n \mathfrak{u}(1)) \\
 \uparrow & & \uparrow \\
 W(\mathfrak{g}) & \xleftarrow{(cs, P)} & W(b^n \mathfrak{u}(1)) \\
 \uparrow & & \uparrow \\
 \text{inv}(\mathfrak{g}) & \xleftarrow{\{P\}} & \text{inv}(b^n \mathfrak{u}(1)) = \text{CE}(b^{n+1} \mathfrak{u}(1))
 \end{array} \tag{481}$$

to be thought of as the universal higher Chern-Simons  $(n+1)$ -bundle with connection on the classifying space of the simply connected Lie group integrating  $\mathfrak{g}$ .

We transgress this to loops by applying the functor  $\text{maps}(-, \Omega^\bullet(S^1))$  from definition 4 to it, which can be thought of as computing for all DGC algebras the DGC algebra of differential forms on the space of maps

from the circle into the space that the original DGCA was the algebra of differential forms of:

$$\begin{array}{ccc}
\text{maps}(\text{CE}(\mathfrak{g}), \Omega^\bullet(S^1)) & \xleftarrow{\text{tg}_{S^1}\mu} & \text{maps}(\text{CE}(b^n\mathfrak{u}(1)), \Omega^\bullet(S^1)) \\
\uparrow & & \uparrow \\
\text{maps}(\text{W}(\mathfrak{g}), \Omega^\bullet(S^1)) & \xleftarrow{\text{tg}_{S^1}(cs, P)} & \text{maps}(\text{W}(b^n\mathfrak{u}(1)), \Omega^\bullet(S^1)) \\
\uparrow & & \uparrow \\
\text{maps}(\text{inv}(\mathfrak{g}), \Omega^\bullet(S^1)) & \xleftarrow{\{\text{tg}_{S^1}P\}} & \text{maps}(\text{CE}(b^{n+1}\mathfrak{u}(1)), \Omega^\bullet(S^1))
\end{array} . \quad (482)$$

We want to think of the result as a  $b^{n-1}\mathfrak{u}(1)$ -bundle. This we can achieve by pulling back along the inclusion

$$\text{CE}(b^{n-1}\mathfrak{u}(1)) \hookrightarrow \text{maps}(\text{CE}(b^n\mathfrak{u}(1)), \Omega^\bullet(S^1)) \quad (483)$$

which comes from the integration current  $\int_{S^1}$  on  $\Omega^\bullet(S^1)$  according to proposition 1.

(This restriction to the integration current can be understood from looking at the basic forms of the loop bundle descent object, which induces *integration without integration* essentially in the sense of [51]. But this we shall not further go into here.)

We now show that the transgressed cocycles  $\text{tg}_{S^1}\mu$  are the familiar cocycles on loop algebras, as appearing for instance in Lemma 1 of [2]. For simplicity of exposition, we shall consider explicitly just the case where  $\mu = \langle \cdot, [\cdot, \cdot] \rangle$  is the canonical 3-cocycle on a Lie algebra with bilinear invariant form  $\langle \cdot, \cdot \rangle$ .

**Proposition 47** *The transgressed cocycle in this case is the 2-cocycle of the Kac-Moody central extension of the loop Lie algebra  $\Omega\mathfrak{g}$*

$$\text{tg}_{S^1}\mu : (f, g) \mapsto \int_{S^1} \langle f(\sigma), g'(\sigma) \rangle d\sigma + (\text{a coboundary}) \quad (484)$$

for all  $f, g \in \Omega\mathfrak{g}$ .

Proof. We compute  $\text{maps}(\text{CE}(\mathfrak{g}), \Omega^\bullet(S^1))$  as before from proposition 1 along the same lines as in the above examples: for  $\{t_a\}$  a basis of  $\mathfrak{g}$  and  $U$  any test domain, a DGCA homomorphism

$$\phi : \text{CE}(\mathfrak{g}) \rightarrow \Omega^\bullet(S^1) \otimes \Omega^\bullet(U) \quad (485)$$

sends

$$\begin{array}{ccc}
t^a & \xrightarrow{\phi} & c^a + A^a\theta \\
\downarrow d_{\text{CE}(\mathfrak{g})} & & \downarrow d_{S^1} + d_U \\
-\frac{1}{2}C^a_{bc}t^b \wedge t^c & \xrightarrow{\phi} & \theta \wedge (\frac{\partial}{\partial \sigma}c^a) + d_U c^a + d_U A^a \wedge \theta \\
& & = -\frac{1}{2}C^a_{bc}c^b \wedge c^c - C^a_{bc}c^b \wedge A^c \wedge \theta
\end{array} . \quad (486)$$

Here  $\theta \in \Omega^1(S^1)$  is the canonical 1-form on  $S^1$  and  $\frac{\partial}{\partial \sigma}$  the canonical vector field; moreover  $c^a \in \Omega^0(S^1) \otimes \Omega^1(U)$  and  $A^a\theta \in \Omega^1(S^1) \otimes \Omega^0(U)$ .

By contracting with  $\delta$ -currents on  $S^1$  we get 1-forms  $c^a(\sigma)$ ,  $\frac{\partial}{\partial \sigma} c^a(\sigma)$  and 0-forms  $A^a(\sigma)$  for all  $\sigma \in S^1$  on  $\text{maps}(\text{CE}(\mathfrak{g}), \Omega^\bullet(S^1))$  satisfying

$$d_{\text{maps}(\dots)} c^a(\sigma) + \frac{1}{2} C^a_{bc} c^b(\sigma) \wedge c^c(\sigma) = 0 \quad (487)$$

and

$$d_{\text{maps}(\dots)} A^a(\sigma) - C^a_{bc} A^b(\sigma) \wedge c^c(\sigma) = \frac{\partial}{\partial \sigma} c^a(\sigma). \quad (488)$$

Notice the last term appearing here, which is the crucial one responsible for the appearance of derivatives in the loop cocycles, as we will see now.

So  $A^a(\sigma)$  (a “field”) is the function on (necessarily flat)  $\mathfrak{g}$ -valued 1-forms on  $S^1$  which sends each such 1-form for its  $t^a$ -component along  $\theta$  at  $\sigma$ , while  $c^a(\sigma)$  (a “ghost”) is the 1-form which sends each tangent vector field to the space of flat  $\mathfrak{g}$ -valued forms to the gauge transformation in  $t^a$  direction which it induces on the given 1-form at  $\sigma \in S^1$ .

Notice that the transgression of our 3-cocycle

$$\mu = \mu_{abc} t^a \wedge t^b \wedge t^c = C_{abc} t^a \wedge t^b \wedge t^c \in H^3(\text{CE}(\mathfrak{g})) \quad (489)$$

is

$$\text{tg}_{S^1} \mu = \int_{S^1} C_{abc} A^a(\sigma) c^b(\sigma) \wedge c^c(\sigma) d\sigma \in \Omega^2(\Omega^1_{\text{flat}}(S^1, \mathfrak{g})). \quad (490)$$

We can rewrite this using the identity

$$d_{\text{maps}(\dots)} \left( \int_{S^1} P_{ab} A^a(\sigma) c^b(\sigma) d\sigma \right) = \int_{S^1} P_{ab} (\partial_\sigma c^a(\sigma)) \wedge c^b(\sigma) + \frac{1}{2} \int_{S^1} C_{abc} A^a(\sigma) c^b(\sigma) \wedge c^c(\sigma), \quad (491)$$

which follows from 487 and 488, as

$$\text{tg}_{S^1} \mu = \int_{S^1} P_{ab} (\partial_\sigma c^a(\sigma)) \wedge c^b(\sigma) + d_{\text{maps}(\dots)}(\dots). \quad (492)$$

Then notice that

- equation 487 is the Chevalley-Eilenberg algebra of the loop algebra  $\Omega\mathfrak{g}$ ;
- the term  $\int_{S^1} P_{ab} (\partial_\sigma c^a(\sigma)) \wedge c^b(\sigma)$  is the familiar 2-cocycle on the loop algebra obtained from transgression of the 3-cocycle  $\mu = \mu_{abc} t^a \wedge t^b \wedge t^c = C_{abc} t^a \wedge t^b \wedge t^c$ .

□

## A Appendix: Explicit formulas for 2-morphisms of $L_\infty$ -algebras

To the best of our knowledge, the only place in the literature where 2-morphisms between 1-morphisms of  $L_\infty$ -algebras have been spelled out in detail is [5], which gives a definition of 2-morphisms for Lie 2-algebras, i.e. for  $L_\infty$ -algebras concentrated in the lowest two degrees. Our definition 17 provides an algorithm for computing 2-morphisms between morphisms of arbitrary (finite dimensional)  $L_\infty$ -algebras. We had already demonstrated in 6.2 one application of that algorithm, showing explicitly how it allows to compute transgression elements (Chern-Simons forms).

For completeness, we demonstrate that the formulas given in [5] for the special case of Lie 2-algebras also follow as a special case from our general definition 17. This is of relevance to our discussion of the String Lie 2-algebra, since the equivalence of its strict version with its weak skeletal version, mentioned in our proposition 25, has been established in [7] using these very formulas. First we quickly recall the relevant

definitions from [5, 7]: A “2-term”  $L_\infty$ -algebra is an  $L_\infty$ -algebra concentrated in the lowest two degrees. A morphism

$$\varphi : \mathfrak{g} \rightarrow \mathfrak{h} \quad (493)$$

of 2-term  $L_\infty$ -algebras  $\mathfrak{g}$  and  $\mathfrak{h}$  is a pair of maps

$$\phi_0 : \mathfrak{g}_1 \rightarrow \mathfrak{h}_1 \quad (494)$$

$$\phi_1 : \mathfrak{g}_2 \rightarrow \mathfrak{h}_2 \quad (495)$$

together with a skew-symmetric map

$$\phi_2 : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{h}_2 \quad (496)$$

satisfying

$$\phi_0(d(h)) = d(\phi_1(h)) \quad (497)$$

as well as

$$d(\phi_2(x, y)) = \phi_0(l_2(x, y)) - l_2(\phi_0(x), \phi_0(y)) \quad (498)$$

$$\phi_2(x, dh) = \phi_1(l_2(x, h)) - l_2(\phi_0(x), \phi_1(h)) \quad (499)$$

and finally

$$\begin{aligned} l_3(\phi_0(x), \phi_0(y), \phi_0(z)) - \phi_1(l_3(x, y, z)) &= \phi_2(x, l_2(y, z)) + \phi_2(y, l_2(z, x)) + \phi_2(z, l_2(x, y)) + \\ &\quad l_2(\phi_0(x), \phi_2(y, z)) + l_2(\phi_0(y), \phi_2(z, x)) + l_2(\phi_0(z), \phi_2(x, y)). \end{aligned}$$

for all  $x, y, z \in \mathfrak{g}_1$  and  $h \in \mathfrak{g}_2$ . This follows directly from the requirement that morphisms of  $L_\infty$ -algebras be homomorphisms of the corresponding codifferential coalgebras, according to definition 13. The not quite so obvious aspect are the analogous formulas for 2-morphisms:

**Definition 43 (Baez-Crans)** A 2-morphism

$$\begin{array}{ccc} & \phi & \\ \curvearrowright & & \curvearrowleft \\ \mathfrak{g} & \begin{array}{c} \parallel \\ \tau \end{array} & \mathfrak{h} \\ \curvearrowleft & & \curvearrowright \\ & \psi & \end{array} \quad (500)$$

of 1-morphisms of 2-term  $L_\infty$ -algebras is a linear map

$$\tau : \mathfrak{g}_1 \rightarrow \mathfrak{h}_2 \quad (501)$$

such that

$$\psi_0 - \phi_0 = t_W \circ \tau \quad (502)$$

$$\psi_1 - \phi_1 = \tau \circ t_v \quad (503)$$

and

$$\phi_2(x, y) - \psi_2(x, y) = l_2(\phi_0(x), \tau(y)) + l_2(\tau(x), \psi_0(y)) - \tau(l_2(x, y)) \quad (504)$$

Notice that  $[d, \tau] := d_{\mathfrak{h}} \circ \tau + \tau \circ d_{\mathfrak{g}}$  and that it restricts to  $d_{\mathfrak{h}} \circ \tau$  on  $\mathfrak{g}_1$  and to  $\tau \circ d_{\mathfrak{g}}$  on  $\mathfrak{g}_2$ .

**Proposition 48** For finite dimensional  $L_\infty$ -algebras, definition 43 is equivalent to the restriction of our definition 17 to 2-term  $L_\infty$ -algebras.

Proof. Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$  and  $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$  be any two 2-term  $L_\infty$ -algebras. Then take

$$\psi, \phi : \mathfrak{g} \rightarrow \mathfrak{h} \quad (505)$$

to be any two  $L_\infty$  morphisms with

$$\mathrm{CE}(\mathfrak{g}) \xleftarrow{\psi^*, \phi^*} \mathrm{CE}(\mathfrak{h}) \quad (506)$$

the corresponding DGCA morphisms. We would like to describe the collection of all 2-morphisms

$$\begin{array}{ccc} & \phi^* & \\ & \downarrow \tau & \\ \mathrm{CE}(\mathfrak{g}) & & \mathrm{CE}(\mathfrak{h}) \\ & \uparrow \psi^* & \end{array} \quad (507)$$

according to definition 17. We do this in terms of a basis. With  $\{t^a\}$  a basis for  $\mathfrak{h}_1$  and  $\{b^i\}$  a basis for  $\mathfrak{h}_2$ , and accordingly  $\{t'^a\}$  and  $\{b'^i\}$  a basis of  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$ , respectively, this comes from a map

$$\tau^* : \mathfrak{h}_1^* \oplus \mathfrak{h}_2^* \oplus \mathfrak{h}_1^*[1] \oplus \mathfrak{h}_2^*[1] \rightarrow \wedge^\bullet \mathfrak{g}^* \quad (508)$$

of degree -1 which acts on these basis elements as

$$\tau^* : b^i \mapsto \tau^i_a t'^a \quad (509)$$

and

$$\tau^* : a^a \mapsto 0 \quad (510)$$

for some coefficients  $\{\tau^i_a\}$ . Now the crucial requirement 149 of definition 17 is that 148 *vanishes* when restricted

$$\begin{array}{ccc} & \mathrm{CE}(\mathfrak{h}) & \\ \phi^* \swarrow & & \nwarrow \\ \mathrm{CE}(\mathfrak{g}) & & \mathrm{W}(\mathfrak{h}) \xleftarrow{\quad} \mathfrak{h}^*[1] \\ \psi^* \searrow & & \swarrow \\ & \mathrm{CE}(\mathfrak{h}) & \end{array} \quad (511)$$

to generators in the shifted copy of the Weil algebra. This implies the following. For  $\tau^*$  to vanish on all  $\sigma t^a$  we find that its value on  $d_{\mathrm{W}(\mathfrak{h})} t^a = -\frac{1}{2} C^a_{bc} t^b \wedge t^c - t^a_i b^i + \sigma t^a$  is fixed to be

$$\tau^* : d_{\mathrm{W}(\mathfrak{g})} t^a \mapsto -t^a_i \tau^i_b t'^b \quad (512)$$

and on  $d_{\mathrm{W}(\mathfrak{h})} b^i = -\alpha^i_{aj} t^a \wedge b^j + c^i$  to be

$$\tau^*(db^i) = \tau^*(-\alpha^i_{aj} t^a \wedge b^j). \quad (513)$$

The last expression needs to be carefully evaluated using formula 145. Doing so we get

$$[d, \tau^*] : t^a \mapsto -t^a_i \tau^i_b t'^b \quad (514)$$

and

$$[d, \tau^*] : b^i \mapsto -\frac{1}{2} \tau^i_a C'^a_{bc} t'^b t'^c - \tau^i_a t'^a_j b'^j + \alpha^i_{aj} \frac{1}{2} (\phi + \psi)^a_b \tau^j_c t'^b t'^c. \quad (515)$$

Then the expression

$$\phi^* - \psi^* = [d, \tau^*] \quad (516)$$

is equivalent to the following ones

$$(\psi^a_b - \phi^a_b)t'^b = t^a_i \tau^i_b t'^b \quad (517)$$

$$(\psi^i_j - \phi^i_j)b'^j = \tau^i_a t'^a_j b'^j \quad (518)$$

$$\frac{1}{2}(\phi^i_{ab} - \psi^i_{ab})t'^a t'^b = -\frac{1}{2}\tau^i_a C'^a_{bc} t'^b t'^c + \alpha^i_{aj} \frac{1}{2}(\phi + \psi)^a_b \tau^j_c t'^b t'^c. \quad (519)$$

The first two equations express the fact that  $\tau$  is a chain homotopy with respect to  $t$  and  $t'$ . The last equation is equivalent to

$$\begin{aligned} \phi_2(x, y) - \psi_2(x, y) &= -\tau([x, y]) + [q(x) + \frac{1}{2}t(\tau(x)), \tau(y)] - [q'(y) - \frac{1}{2}t(\tau(y)), \tau(x)] \\ &= -\tau([x, y]) + [q(x), \tau(y)] + [\tau(x), q'(y)] \end{aligned} \quad (520)$$

This is indeed the Baez-Crans condition on a 2-morphism.  $\square$

## Acknowledgements

We thank Danny Stevenson for many helpful comments. Among other things, the discussion of the cohomology of the string Lie 2-algebra has been greatly influenced by conversation with him. We acknowledge stimulating remarks by John Baez on characteristic classes of string bundles. H.S. thanks Matthew Ando and U.S. thanks Stephan Stolz and Peter Teichner for discussions at an early stage of this work.

U.S. thankfully acknowledges invitations to the University of Oxford by Nigel Hitchin; to the conference “Lie algebroids and Lie groupoids in differential geometry” in Sheffield by Kirill Mackenzie and Ieke Moerdijk; to the Erwin Schrödinger institute in the context of the program “Poisson  $\sigma$ -models, Lie algebroids, deformations and higher analogues”; to “Categories in Geometry and Physics” by Zoran Škoda and Igor Baković; and to Yale University which led to collaboration with H.S. and to useful discussions with Mikhail Kapranov. U.S. thanks Todd Trimble for discussion of DGCAs, and their relation to smooth spaces, thanks Mathieu Dupont and Larry Breen for discussion about weak cokernels and thanks Simon Willerton and Bruce Bartlett for helpful discussion of the notion of transgression. H.S. thanks Akira Asada and Katsuhiko Kuribayashi for useful correspondence on their work on (higher) string structures.

We thank the contributors to the weblog *The n-Category Café* for much interesting, often very helpful and sometimes outright invaluable discussion concerning the ideas presented here and plenty of related issues.

We have gratefully received useful comments on earlier versions of this document from Gregory Ginot and David Roberts. Evan Jenkins and Todd Trimble have provided useful help with references to literature on smooth function algebras.

While this article was being written, the preprints [76], [54] and [35] (based on [36]) appeared, which are closely related to our discussion here in that they address the issues of characteristic classes and extensions of  $n$ -bundles in one way or another. We are thankful to Amnon Yekutieli and to Gregory Ginot for pointing out their work to us and for further discussion.

Finally, we thank Bertfried Fauser and the organizers of the conference “Recent developments in QFT” in Leipzig.

U. S. thankfully acknowledges financial support from the DAAD German-Croatian bilateral project “Nonabelian cohomology and applications”.



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